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A GENERALIZED VORTEX THEORY OF THE SCREW PROPELLER

AND ITS APPLICATION

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By Hans Reissner

SUMMARY

The vortex theory as presented by the author in earlier papers has been extended to permit the solution of the following problems:

- (1) The investigation of the relation between thrust and torque distribution and energy loss as given by the induction of helical vortex sheets and by the parasite drag;
- (2) The checking of the theorem of Betz of the rigidly behaving helical vortex sheet of minimum induced energy loss;
- (3) The extension of the theory of the screw propeller of minimum energy loss for the inclusion of parasite-drag distribution along the blades.

A simple system of diagrams has been developed to systematize the design of airplane propellers for a wide range of practical application. Several typical diagrams are presented to illustrate the method.

INTRODUCTION

In earlier papers (references 1 and 2), the author presented a theory that followed the conception presented by Betz and Prandtl in 1919 (reference 3) of the helical vortex sheets left behind by the trailing edges of the propeller blades.

The relation between the thrust and the torque distribution and energy loss have been given in a somewhat new and more precise form by rational superposition of the

influence of potential flow and of friction. The theory of Betz was proved more rigorously by extending a method of proof used by Fuchs (reference 4) for the straight-moving airfoil to the rotating airfoil.

A new theorem for the screw propeller of induced and parasite drag energy loss can be stated by a very simple formula giving the induced inflow angle change of minimum total resistance (or energy loss) as affected by an arbitrary distribution of parasite drag along the blade.

A method of computing the induction of rigid or non-rigid vortex sheets entirely different from the method of Goldstein has been followed. The results of this method, as far as they correspond to rigid sheets and no parasite drag, shall be compared with the results of Goldstein (reference 5). In order to derive a system of tables and curves for the range of engine powers, speeds, air densities, and relative inflow for the design of airplane propellers, a great amount of work is necessary and all the charts have not been completed. It was necessary to develop the circulation function upon which all the blade dimensions depend into a power series along the blades, satisfying certain boundary conditions at the tip and the root. The values of the constants were determined by making a minimum the mean square of the deviation from the prescribed inflow angle change of minimum total energy loss. Such a mean square can be defined in different ways, which either emphasize the correctness near the tip or near the root.

About 200 tabulation sheets and 86 diagrams have been prepared, embracing tip speed-advance ratios from 1 to 5 and blade numbers from 2 to 4. These diagrams allow immediate application for the ratios 3 to 5 but, for the ratios 1 and 2 giving high flight velocity, more accurate values of the Bessel functions appearing in the integrals will have to be determined for the middle part of the blades. Twenty-six of the diagrams are reproduced as sample sheets in this paper.

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NOTATION

- Q , torque acting on propeller either directly or through a gear from the engine.
 T , thrust of propeller.
 E , total ^{rate of} loss of energy ($Q\omega - Tw$).
 u , radial velocity.
 ω , angular velocity of propeller ($\frac{2\pi}{60} n$).
 n , propeller speed, revolutions per minute.
 w , axial velocity of air relative to plane of rotation of propeller.
 V , resultant inflow velocity ($\sqrt{w^2 + \omega^2 r^2}$).
 V^* , induced velocity.
 r , radius of blade element.
 r_a , radius of blade tip.
 μ , density of medium (air) in which propeller is working.
 c , blade width.
 l , number of blades.
 p , dimensionless radial abscissa or reciprocal advance per unit of circumferential travel.
 α , angle with plane of rotation of undisturbed relative resultant inflow ($\cot^{-1} p$; $\bar{p} = \cot \alpha$).
 C_L , lift coefficient.
 C_D , parasite-drag coefficient.
 C_Q , torque coefficient ($\frac{Q\omega^3}{\mu w^5}$).
 C_T , thrust coefficient ($\frac{Tw^2}{\mu w^4}$).

C_E , energy-loss coefficient $\left(\frac{E \omega^2}{\mu w^5} \right)$.

z , axial coordinate.

Γ , circulation.

Γ^* , circulation coefficient $\left(\Gamma \frac{\omega i}{w^2} \right)$.

δ , angle of parasite drag ($\delta = \cotan^{-1} C_L/C_D$).

i , change of angle α of undisturbed inflow produced by the induction of the helical vortex sheets.

FUNDAMENTAL PROPOSITIONS

A dimensionless radial number

$$\rho = \omega r/w \quad (1)$$

According to figure 1:

$$dL = C_L \mu \frac{v^2}{2} dA \quad (2)$$

$$dD = C_D \mu \frac{v^2}{2} dA \quad (3)$$

$$E = Q \omega - T w \quad (4)$$

Considering (figs. 1 and 2) that the resultant elementary lift dL acting on the elementary area $dA = bdr$ of a blade is perpendicular to the incidence line of the flow as it is changed by the induced angle i and the drag angle δ , the thrust, the torque, and the energy loss can be expressed by the following integrals:

$$Q = b \int_0^{r_a} dL \sin (\alpha + i + \delta) r \quad (5)$$

$$T = b \int_0^{r_a} dL \cos (\alpha + i + \delta) \quad (6)$$

$$E = b \int_0^{r_a} dL w \frac{\sin (i + \delta)}{\sin \alpha} \quad (4a)$$

For the sections used in propeller design and for the practical angles of attack between the direction of no lift and the relative inflow direction, the angle of parasite drag changes slowly going from the maximum angles used toward the zero angle of attack but increases rapidly in the other direction toward the angle of maximum lift.

The lift dL that the air current exerts on a blade element cdr shall be expressed by the circulation Γ produced by the helical vortex sheets originating at the trailing edges of all the blades in the following manner: Equation (2) becomes in this case

$$dL = C_L \frac{\mu}{2} V^2 cdr \quad (2a)$$

where

$$V^2 = \omega^2 r^2 + w^2 = w^2 (\rho^2 + 1) = w^2 (1 + \cot^2 \alpha) = \frac{w^2}{\sin^2 \alpha} \quad (7)$$

and where, following Wieselsberger's relation, the lift coefficient C_L shall be split in the ideal (frictionless) part and the deduction caused by the parasite drag. This simple superposition of the two effects of potential flow and of friction has been proved to be sufficiently accurate and very useful by experience and practice. Put

$$C_L = C_{L,0} (1 - k\delta) \quad (8)$$

with the experimental constant k having a value of about 7.5.

These relations must now be compared with the Kutta-Joukowski-Prandtl relation between circulation Γ and lift

$$dL_0 = \mu \Gamma V dr \quad (9)$$

where dL_0 is the "ideal" lift element; that is, the lift element for an airfoil without parasite drag.

Comparing now equations (2a), (8), and (9), obtain

$$C_{L,0} c V = 2 \Gamma$$

and

$$dL = \mu \Gamma V dr (1 - k\delta) = \mu \frac{w^2}{\omega} \Gamma \frac{d\rho}{\sin \alpha} (1 - k\delta) \quad (10)$$

Introducing then a dimensionless circulation coefficient by putting

$$\Gamma = \Gamma^* \frac{w^2}{\omega l} \quad (11)$$

remembering that, from equation (1),

$$\sin \alpha = \frac{1}{\sqrt{1 + \rho^2}}, \quad \cos \alpha = \frac{\rho}{\sqrt{1 + \rho^2}}$$

and taking into account that Γ^* and $i + \delta$ are small quantities of which products and power only up to the second power need be retained, it is possible to write the relations (5), (6), and (4a) in dimensionless form, as follows:

Dimensionless force and energy coefficients may first be introduced:

$$C_Q = \frac{Q\omega^3}{\mu w^5} \quad (12)$$

$$C_T = \frac{T\omega^2}{\mu w^4} \quad (13)$$

$$C_E = \frac{E\omega^2}{\mu w^5} \quad (14)$$

Then the relations (5), (6), and (4a) appear in the following pure-number expressions:

$$C_Q = \int_{\rho_1}^{\rho_a} d\rho \rho \Gamma^* [1 + \rho(i + \delta)](1 - k\delta) \quad (5a)$$

$$C_T = \int_{\rho_1}^{\rho_a} d\rho \Gamma^* [\rho - (i + \delta)](1 - k\delta) \quad (6a)$$

$$C_E = \int_{\rho_1}^{\rho_a} d\rho \Gamma^* (i + \delta)(1 + \rho^2)(1 - k\delta) \quad (4b)$$

In references 1 and 2, three important problems of the vortex-sheet theory of the propeller are stated: namely,

- (a) To determine the distribution of induced-angle changes i of the inflow along the blade radii produced by given distributions of circulation coefficient Γ^* (or vortex-distribution coefficient $d\Gamma^*/dp$).
- (b) To determine the distribution of circulation coefficient Γ^* following from a given distribution of blade width, blade section, and blade-angle setting.
- (c) To determine those distributions of circulation coefficient Γ^* and tip radial numbers $\rho_a = \omega r_a / w$ that give the minimum energy-loss coefficient C_E if the value either of the torque coefficient C_Q or of the thrust coefficient C_T is prescribed.

In references 1 and 2, methods have been indicated to solve these problems. This paper shall consider only problems (a) and (c). It shall bring a new proof and an extension of the Betz theorem of the rigid vortex sheet and shall work out one of the indicated methods as described in detail in the following sections.

INFLOW ANGLE CHANGE INDUCTION i FOR MINIMUM ENERGY LOSS

Before the discussion proper, a remark shall be made about a reciprocal theorem in potential theory and its application on the determination of distributions of minimum induced resistance.

R. Fuchs has proved, by means of this reciprocal theorem, M. Munk's theorem of the elliptic distribution of circulation and of the rigidly moving vortex sheet for airfoils in translation and has suggested that it also might be applied to the problem of airfoils rotating in a relative air current and, in this way, the induced resistance or energy loss of the rigid helical vortex sheet investigated. (See reference 4.)

The following discussion shows that the reciprocal

theorem leads to the result that the generalization of Munk's theorem to the helical vortex sheet is justified up to small quantities of the second order and can even be further generalized to include the parasite drag.

The reciprocal theorem (due to Gauss) will first be given in the language of potential theory and then translated into the language of hydrodynamics.

Given a closed surface A and two potential functions φ_1 and φ_2 continuous with continuous derivatives, the reciprocal theorem of Gauss can then be pronounced in the form

$$\oint dA \varphi_1 \frac{\partial \varphi_2}{\partial n} = \oint dA \varphi_2 \frac{\partial \varphi_1}{\partial n} \quad (15)$$

where the derivatives are to be taken in respect to the normal of the surface taken everywhere to the inside (or everywhere to the outside).

If now (fig. 2) this surface is given by two of the helical vortex sheets, two plane sections parallel to the plane of rotation and a cylindrical surface coaxial to the axis of rotation at a sufficiently great distance from it, then only the derivatives at the vortex sheets give a contribution to the integral because: By the supposition of helical symmetry, the terms of the two sectional planes cancel each other; the terms on the remote cylindrical surface disappear, decreasing more quickly than the reciprocal square of the radial distance; and the terms on the helical surfaces also cancel each other because, on account of helical symmetry, the potentials are equal and the derivatives are of opposite sign on account of the definition of the positive sign of the normal.

Observing then that

$$\begin{aligned} dA &= dz \, dr \, \frac{V}{w} \\ &= dz \, dr \, \sqrt{1 + \rho^2} \end{aligned}$$

and that $\partial\varphi/\partial n$ is the velocity induced by the vortex sheet, that is, (see fig. 1)

$$\frac{\partial\varphi}{\partial n} = v^* = iV = i\omega \sqrt{1 + \rho^2}$$

the reciprocal theorem can be written for this special case in the form

$$\int \Gamma^*_1 i_2 (1 + \rho^2) d\rho = \int \Gamma^*_2 i_1 (1 + \rho^2) d\rho \quad (15a)$$

the integral taken only over the discontinuities of one helical discontinuity (or vortex) sheet.

Comparing with (4b) and assuming for a moment the parasite-drag angle $\delta = 0$, this relation gives a reciprocal theorem for the energy loss corresponding to one kind of circulation function Γ_1 (or Γ^*_1 written dimensionless) and its corresponding flow deviation i_1 and another kind of circulation Γ_2 (Γ^*_2) and its corresponding flow deviation i_2 : namely,

$$C_{E,1,2} = C_{E,2,1} \quad (15b)$$

The idea of the proof of Fuchs is to consider first the energy loss produced by the difference of two circulation-deviation distributions $\Gamma^*_1, i_1, \Gamma^*_2, i_2$: namely,

$$\begin{aligned} C_{E,1-2} &= \int d\rho (\Gamma^*_1 - \Gamma^*_2) (i_1 - i_2) (1 + \rho^2) \\ &= C_{E,1} + C_{E,2} - \int d\rho (\Gamma^*_1 i_2 + \Gamma^*_2 i_1) (1 + \rho^2) \end{aligned}$$

and, according to (15a),

$$C_{E,1-2} = C_{E,1} + C_{E,2} - 2 \int d\rho (1 + \rho^2) \Gamma^*_2 i_1$$

Now for a rigid helical sheet, λ being an unknown constant, the induced velocity V^*_r and the induced angle i_r would be

$$\frac{V^*_r}{V} = i_r = \lambda \frac{\rho}{1 + \rho^2} \quad (16)$$

Assuming $i_r = i_1$ and inserting this value into the last equation of the energy loss produced by the differences, obtain

$$C_{E,1-2} = C_{E,1} + C_{E,2} - 2\lambda \int d\rho \Gamma^*_2 \quad (17)$$

As either the value of torque Q or of thrust T must be prescribed (which one does not make any difference), prescribe the torque of the engine according to (5a) and, interested for the moment only in the energy loss by the induced resistance but not by drag, the condition of keeping the torque constant is:

$$C_Q = \int_{\rho_1}^{\rho_a} d\rho \rho \Gamma^*_1 (1 + \rho i_1) = \int_{\rho_1}^{\rho_a} d\rho \rho \Gamma^*_2 (1 + \rho i_2)$$

and, if again for i_1 the value appertaining to the rigid vortex sheet is introduced, then:

$$\int d\rho \rho \Gamma^*_1 \left(1 + \lambda \frac{\rho^2}{1 + \rho^2} \right) = \int d\rho \rho \Gamma^*_2 (1 + \rho i_2)$$

or

$$\begin{aligned} \lambda \int d\rho \rho \Gamma^*_2 &= \lambda \int d\rho \rho \Gamma^*_1 + \lambda \int d\rho \rho^2 \left(\Gamma^*_1 \frac{\lambda \rho}{1 + \rho^2} - \Gamma^*_2 i_2 \right) \\ &= C_{E,1} + \lambda \int d\rho \rho^2 \left(\Gamma^*_1 \frac{\lambda \rho}{1 + \rho^2} - \Gamma^*_2 i_2 \right) \end{aligned}$$

Inserting this expression in (17), this equation leads to

$$C_{E,2} = C_{E,1-2} + C_{E,1} + 2\lambda \int d\rho \rho^2 \left(\Gamma^*_1 \frac{\lambda \rho}{1 + \rho^2} - \Gamma^*_2 i_2 \right) \quad (18)$$

Now the last integral is a small quantity of the third order, if Γ^* , i , and λ are all of the first order of small quantities, and therefore

$$C_{E,2} = C_{E,1} + C_{E,1-2} \quad (18a)$$

and Betz's generalization of Munk's theorem, to attain the minimum induced energy loss for a given torque or a given thrust, is proved to be correct by the method of Fuchs.

This significance of (18a) results from the fact that the energy loss $C_{E,1-2}$ must be positive and therefore $C_{E,1}$ must be smaller than any other possible loss $C_{E,2}$ of energy.

In order now to extend the minimum condition to the total-energy loss including the parasite drag given by the drag angle δ , if either torque or thrust of the propeller (also including the effect of the drag) is prescribed, the energy loss corresponding to the difference of two circulation functions Γ^*_1 , Γ^*_2 and to the difference of

the corresponding deviation angles i must be expressed in the general form containing the drag angle δ .

This expression is given by

$$\begin{aligned} C_{E,1-2} &= \int d\rho(1 + \rho^2)(\Gamma^*_1 - \Gamma^*_2)[i_1 + \delta - (i_2 + \delta)](1 - k\delta) \\ &= C_{E,1} + C_{E,2} - \int d\rho(1 + \rho^2)[\Gamma^*_1(i_2 + \delta) + \Gamma^*_2(i_1 + \delta)](1 - k\delta) \end{aligned}$$

Now the potential part of the flow satisfies again the condition (15a)

$$\int d\rho(1 + \rho^2)\Gamma^*_1 i_2 = \int d\rho(1 + \rho^2)\Gamma^*_2 i_1$$

Therefore, neglecting small quantities of the third order like $\Gamma^* i \delta$ against the quantities of the second order like $\Gamma^* i$ or $\Gamma^* \delta$:

$$C_{E,1-2} = C_{E,1} + C_{E,2} - 2 \int d\rho(1 + \rho^2)\Gamma^*_2 \left(i_1 + \frac{\delta}{2}\right) - \int d\rho(1 + \rho^2)\Gamma^*_1 \delta$$

Putting then

$$i_1 + \frac{\delta}{2} = \lambda \frac{\rho}{1 + \rho^2} \quad (16a)$$

the last equation becomes

$$C_{E,1-2} = C_{E,1} + C_{E,2} - 2\lambda \int d\rho \rho \Gamma^*_2 - \int d\rho(1 + \rho^2)\Gamma^*_1 \delta \quad (18b)$$

The third term on the right-hand side can now be transformed by the condition of prescribed torque, according to (5a):

$$\begin{aligned} C_Q &= \int d\rho \rho \Gamma^*_1 \left[1 + \rho \left(\lambda \frac{\rho}{1 + \rho^2} + \frac{\delta}{2} \right) \right] (1 - k\delta) \\ &= \int d\rho \rho \Gamma^*_2 [1 + \rho (i_2 + \delta)] (1 - k\delta) \end{aligned}$$

This equation, if small quantities of higher order than the second are again neglected, furnishes:

$$2\lambda \int d\rho \rho \Gamma^*_2 = 2\lambda \int d\rho \rho \Gamma^*_1$$

On the other hand,

$$\begin{aligned}
 2C_{E,1} &= 2 \int dp \Gamma^*_1 (1 + \rho^2) \left(\lambda \frac{\rho}{1 + \rho^2} + \frac{\delta}{2} \right) \\
 &= 2 \lambda \int dp \rho \Gamma^*_1 + 2 \int dp \Gamma^*_1 (1 + \rho^2) \frac{\delta}{2}
 \end{aligned}$$

Putting this expression into (18b), there arises finally again the result

$$C_{E,2} = C_{E,1} + C_{E,1-2} \quad (18a)$$

which result signifies an extension of Betz's theorem from ideal propellers to the actual propeller having parasite drag as follows:

The propeller of minimum loss of energy, if torque or thrust is prescribed, must leave behind its trailing blade edges vortex sheets that will produce perpendicular to them induced inflow-angle changes (i) equal to the difference between the inflow-angle changes $\lambda \frac{\rho}{1 + \rho^2}$ of rigid helical surfaces and the half of the angles δ of parasite drag of the blade sections.

In this theorem, the factor λ remains indeterminate until the relation between the circulation Γ^* and the induced angle deviation i (or induced velocity $V^* = iV$) has been developed; whereas the drag angle δ , although not varying much for the most important range of angles of attack, must be chosen from experiment according to the shape and the smoothness of each section of the blade.

DETERMINATION OF DISTRIBUTION OF CIRCULATION

(THRUST AND TORQUE)

In order to apply the results arrived at in the foregoing pages, it is necessary to answer the question: Which circulation distribution Γ^* would be necessary to produce a desired induced change i of inflow angle?

This question has not been answered directly until now, except for the inflow angle change of the rigid vortex sheet by S. Goldstein (reference 5), but the inverse prob-

lem, to determine i from an arbitrarily given circulation Γ^* , is in principle solved by the Biot-Savart integral.

In references 1 and 2, there has been derived for this purpose a formula different from the Biot-Savart integral, which avoids the three-dimensional space integrations and the inconveniences of apparent singularities in the Biot-Savart solution and gives the resultant induced angle change i as the sum of three terms.

The first term, which is independent of the number of blades (that is, for an infinite number of infinitely small blades), signifies the mean value of i on each arc between each pair of 1 blades and can be called the self-induction i_s of the circulation at a blade point on this point.

The second term represents the effect i_i of the circulation existing on the radius inside of a point, on this point, using an infinite series of integrals containing the circulation distribution in this interval and dependent on the number of blades.

The third term in an analogous manner represents the effect i_e on this point of the circulation existing on the part of the radius outside of it.

Introducing the Bessel functions of imaginary argument of the first and second kinds $I_{n1}(n1\rho)$ and $K_{n1}(n1\rho)$ written in Watson's notation as real functions and writing for abbreviation

$$n1 = m, \quad I_{n1}(n1\rho) \equiv I_m, \quad K_{n1}(n1\rho) \equiv K_m$$

the formula may be written:

$$i(\rho) = \frac{\Gamma^*}{4\pi\rho} - \frac{1}{2\pi\rho} \Sigma \left[K_m(\rho) \int_0^\rho I_m(\xi) d\left(\xi \frac{d\Gamma^*}{d\xi}\right) + I_m(\rho) \int_\rho^{\rho_a} K_m(\xi) d\left(\xi \frac{d\Gamma^*}{d\xi}\right) \right] \quad (20)$$

This formula covers the case where the induced velocities V^* or the angles i remain finite at the axis and at infinite radial distance.

Anyhow it may be useful for applications to give also the formula for the case where a hub body, as in figure

(2a) makes the radial velocity u zero for $r = r_1$, that is, for $\rho = \rho_1$.

In equations (4), (7), and (9) of reference 1, the two integration constants $f_n^{(1)}$ and $f_n^{(2)}$ have to be determined for the assumption of a hub body, so that:

- (a) for $\rho = \infty$, it is required that $u^{(0)} + u^{(1)} = 0$, $v^{(0)} + v^{(1)} = 0$, $w^{(0)} + w^{(1)} = 0$, and this relation is again accomplished by putting

$$f_n^{(1)} = - \int_{\rho_1}^{\rho_a} K_m(\xi) d\left(\xi \frac{d\Gamma^*}{d\xi}\right)$$

- (b) for $\rho = \rho_1$, it is required that $u^{(0)} + u^{(1)} = 0$, $v^{(0)} + v^{(1)}$ finite, $w^{(0)} + w^{(1)}$ finite and this expression brings with it

$$f_n^{(2)} = 0, \left(\frac{d\Gamma^*}{d\rho}\right)_{\rho=\rho_1} = 0, (\Gamma^*)_{\rho=\rho_1} = 0 \quad (21)$$

These conditions change formula (20) into

$$i(\rho) = \frac{\Gamma^*}{4\pi\rho} - \frac{1}{2\pi\rho} \Sigma \left[K_m(\rho) \int_{\rho_1}^{\rho} I_m(\xi) d\left(\xi \frac{d\Gamma^*}{d\xi}\right) + I_m(\rho) \int_{\rho}^{\rho_a} K_m(\xi) d\left(\xi \frac{d\Gamma^*}{d\xi}\right) \right] \quad (20a)$$

provided that conditions (21) are observed for the choice of the function Γ^* .

This fact would imply that, for every change of hub diameter (change of ρ_1), another function Γ^* would have to be employed. Yet, if ρ_1 is sufficiently small in relation to ρ_a and if only Γ^* and $d\Gamma^*/d\rho$ are sufficiently small at the point ρ_1 , then conditions (21) can be neglected in view of the fact that, on account of the fillets between the hub spinner and the blade, conditions (21) are actually only approximately valid.

If now the problem is advanced to design the 1

blades with such widths, cross sections, and angle positions that, for a given speed of the airplane w (corrected for the influence of hull), an angular velocity of the shaft ω (speed of the engine), and a given torque Q , thrust is a maximum or, what is the same, the energy loss is a minimum, then the distribution of circulation Γ^* must be chosen according to the law (16a) for the inflow angle deviation i previously derived: namely,

$$i = \lambda \frac{\rho}{1 + \rho^2} - \frac{\delta}{2} \quad (16a)$$

where the δ 's are the drag angles of the cross sections of the blade, generally increasing with thickness, that is, from the tip to the root of the blade; and λ is a constant to be determined to give the required torque Q according to (5a).

The relation (16a) must then be introduced into (20a) so that the following determining equation for Γ^* arises:

$$\Gamma^* = 4\pi\rho \left(\lambda \frac{\rho}{1 + \rho^2} - \frac{\delta}{2} \right) + 2\Sigma \left[K_m(\rho) \int_{\rho_1}^{\rho} I_m(\xi) d\left(\xi \frac{d\Gamma^*}{d\xi} \right) + I_m(\rho) \int_{\rho}^{\rho_2} K_m(\xi) d\left(\xi \frac{d\Gamma^*}{d\xi} \right) \right] \quad (22)$$

This equation looks like a nonhomogeneous integral equation of the second kind for the special case where, under the integral, the function to be determined appears as a derivative. Trying to differentiate this equation in such a manner that the left-hand side is transformed into

$$\frac{d}{d\rho} \left(\rho \frac{d\Gamma^*}{d\rho} \right) \quad \text{and then introducing this expression as a new}$$

function y to arrive at the customary form of the integral equation, one finds that, following a property of the Bessel function, the series splits into two divergent series. It has been proved, however, that the series is convergent before this transformation, so that this transformation does not lead to a useful method. It shows only that the induced velocity or inflow-angle change i can be determined as a summation of the vortex distribution

$d\Gamma^*/d\rho$ but not as the summation of the effects of the circulation (Γ^*) distribution.

The solution of (22) has been effected in the following way: Assuming for Γ^* a polynomial series in ρ with, at first, arbitrary coefficients but satisfying the condition that each polynomial term Γ^*_p is zero at the inner and the outer boundaries ρ_i and ρ_a and that the derivative $d\Gamma^*/d\rho$ is zero at $\rho = \rho_i$, it is assumed for the special case $\rho_i = 0$:

$$\Gamma^* = \sum_{p=0}^{p=p_1} A_p \Gamma^*_p, \quad \text{where } \Gamma^*_p = \rho^{p+2} (\rho_a - \rho) \quad (23)$$

The angle deviations i produced by this circulation function have been computed by (20a), as will be described in detail, in the form

$$i = \sum_{p=0}^{p=p_1} A_p i_p \quad (23a)$$

where the i_p 's are now known as functions of ρ and ρ_a .

Comparing now (16a) and (23a), the constants are to be determined by the equation

$$\sum_{p=0}^{p=\infty} A_p i_p = \left(\lambda \frac{\rho}{1 + \rho^2} - \frac{\delta}{2} \right)_{\rho_a > \rho > \rho_i} \quad (24)$$

with the condition for prescribed torque

$$C_Q = \int_{\rho_i}^{\rho_a} d\rho \rho \sum (A_p \Gamma^*_p) \left[1 + \rho \left(\lambda \frac{\rho}{1 + \rho^2} + \frac{\delta}{2} \right) \right] (1 - k\delta) \quad (5b)$$

This integral is easily calculated as soon as the function δ is prescribed and gives the value of the constant λ .

In order to find the best representation of the right side of (24) for a limited number of terms p , the mean square of the deviation from the ideal function, that is, from the right side of (24), shall be made a minimum in such a way that to the deviations from their true values is allotted greater weight toward the tip of the blade, increasing with the area covered in rotation.

This method furnishes the following condition for the coefficients A_p

$$\int_{\rho_i}^{\rho_a} d\rho \rho \left[\lambda \frac{\rho}{1 + \rho^2} - \frac{\delta}{2} - \sum_{p=0}^{p=p_1} A_p i_p(\rho) \right]^2 = \text{minimum}$$

The differentiations in respect to each of all the coefficients A_p gives the following set of equations for the A_p 's:

$$\sum_{p=0}^{p=p_1} A_p \int_{\rho_i}^{\rho_a} d\rho \rho i_n(\rho) i_p(\rho) = \lambda \int_{\rho_i}^{\rho_a} \frac{d\rho \rho^2}{1 + \rho^2} i_n(\rho) - \frac{1}{2} \int_{\rho_i}^{\rho_a} d\rho \rho \delta(\rho) i_n(\rho) \quad (24a)$$

In each of these equations appears one angle i_n . The number of these equations is equal to the number p_1 of polynomials Γ^*_p chosen to design the blade because n runs from $n = 0$ to $n = p_1$.

The functions i_p and i_n found by (20a) and (24a) express all the constants A_p by the constant λ . The constant λ must then be determined by means of the prescribed value of the torque coefficient C_Q , that is, by introducing the A_p, λ relations from (24a) into (5b), which then becomes a quadratic equation for λ . Of the two roots of this equation, only the small and positive one has a physical meaning.

CALCULATION OF ANGLES i_p INDUCED BY

THE CIRCULATION TERMS Γ^*_p

For the procedure described, it is necessary to compute the functions $i_p(\rho)$ by means of (20a). As the Bessel functions of imaginary argument $I_{n1}(n! \rho) \approx I_m$ and $K_{n1}(n! \rho) \approx K_m$ are not tabulated as far as they are needed here, it had already been suggested by the author (references 1 and 2) that the asymptotic expressions of Nicholson (reference 6) be used. For the purpose of this problem, the first terms of these expressions are sufficiently ac-

curate down to a value of a radial distance number $\rho = 0.6$; which, for $n! = 2$, gives an error of 2 percent and, for the next terms with $n! = 4$ and $n! = 6$, gives errors less than 0.1 percent. In a supplementary section, it is shown how values for points nearer the origin are determined by the classical power series of the Bessel functions and how the behavior of the curves, to be computed, approaching the origin, is ascertained.

The Nicholson formulas for the Bessel functions of imaginary argument as used in references 1 and 2 have been written as follows:

Introducing as a new independent variable

$$u = \sqrt{1 + \rho^2}, \quad v = \sqrt{1 + \xi^2}, \quad \xi = \sqrt{v^2 - 1} \quad (25)$$

then Nicholson's formula can be written:

$$I_m(\rho) = \frac{1}{\sqrt{2m\pi}} \frac{1}{\sqrt{u}} \left(e^u \sqrt{\frac{u-1}{u+1}} \right)^m$$

$$K_m(\rho) = \sqrt{\frac{\pi}{2m}} \frac{1}{\sqrt{u}} \left(e^u \sqrt{\frac{u-1}{u+1}} \right)^{-m}$$

and, introducing the development (23) for Γ^* ,

$$\Gamma_p^* = \rho^{p+2} (\rho_a - \rho)$$

$$\frac{d}{d\xi} \left(\xi \frac{d\Gamma_p^*}{d\xi} \right) = \xi^{p+1} \left[(p+2)^2 \rho_a - (p+3)^2 \xi \right]$$

the term i_p corresponding to the contribution of Γ_p^* can be written (using (25)):

$$i_p = \frac{1}{4\pi} \rho^{p+1} (\rho_a - \rho)$$

$$- \frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \frac{1}{\sqrt{u(u^2-1)}} e^{-mu} \left(\frac{u-1}{u+1} \right)^{-\frac{m}{2}} \int_{\rho_i}^{\rho} dv \sqrt{v} e^{mv} \frac{(v-1)^{\frac{m+p}{2}}}{(v+1)^{\frac{m-p}{2}}} \left[(p+2)^2 \rho_a - (p+3)^2 \sqrt{v^2-1} \right] \right. \\ \left. + \frac{1}{\sqrt{u(u^2-1)}} e^{mu} \left(\frac{u-1}{u+1} \right)^{\frac{m}{2}} \int_{\rho}^{\rho_a} dv \sqrt{v} e^{-mv} \frac{(v-1)^{\frac{m+p}{2}}}{(v+1)^{\frac{m-p}{2}}} \left[(p+2)^2 \rho_a - (p+3)^2 \sqrt{v^2-1} \right] \right\} \quad (20b)$$

In expression (20b), there appear four integrals. Each of the integrands has been computed for the values of $m = 2, 4, 6, 8$, and 10 and for values of $p = 0, 1$, and 2.

The calculation showed, as is to be seen, in figures 3, 4, and 5, that the integrands increased so rapidly with the abscissa v for I_m and $v_a - v$ for K_m that it was necessary to change the scale very often in plotting them along the abscissa.

This change was particularly necessary in order to be able to draw the integral curves, that is, in order to determine the areas for a sufficiently great number of

abscissa values $\xi = \sqrt{v^2 - 1}$ or $v = \sqrt{\xi^2 + 1}$.

Each of the integrals in (20b) was represented by two curves corresponding to the two terms under the integral signs in (20b) from which the different curves resulted.

In this way for the induction from the inside, the sets of curves for the different tip radii numbers $u_a = 2, 3, 4$, and 5 in figures 6 and 7 and, for the induction from the outside, the sets of curves in figures 8 and 9 were drawn for the different tip radii numbers u_a , the different orders in p , and the different orders in m .

For the induction at each one of the points $u = \sqrt{\rho^2 + 1}$ between $u = 1$ and $u = u_a$, the integral as well as from the outside as from the inside had to be multiplied by the value of $K_m(\rho)$ and $I_m(\rho)$, respectively, at the point considered, as equation (20b) shows.

This task was rather laborious by tabulation as well as by drawing, but nothing simpler could be found.

Having then determined the angle inflow induction i_p from inside and outside for a certain tip radius number

$u_a = \sqrt{\rho_a^2 + 1}$ and a number of points between $u = 1$ ($\rho = 0$) and u_a , both inductions were superposed in figures 10 to 16 so that a graphical picture of the total induction for every p and for every m was gained.

Finally, for each p separately but for all the

values of m that have been computed, the induction curves have been added and a remainder of the m series has been added by utilizing the fact proved elsewhere that the convergence of the series in m approaches the series

$$\sum \frac{1}{m^2} \text{ with sufficient approximation.}$$

These resulting curves are shown in figures 17 to 24. All these curves do not contain the first term of "self" induction on the right side of (20b), which is, very simply, to be calculated and added to the "distant" induction curves.

It may be remarked here that, for reasons of being able to compute (20b) graphically, the functions chosen in (23) have no vertical tangent at the point $p = p_a$, as is the case for elliptical distribution.

Such a proposition would have made $\frac{d}{d\xi} \left(\xi \frac{d\Gamma^*}{d\xi} \right)$ infinite. Yet the inclination of the tangent to the curve Γ^* can be made as great as desired by even a single term of (23) if the condition of minimum energy loss would demand it.

For the following computation; two blades ($l = 2$) have been assumed but, in the future computations, many of the integrals can be used for other blade numbers as only the order number $m = n!$ ($n = 1, 2, 3 \dots$) is decisive.

In references 1 and 2, it has been shown that the convergence of the series in (20b) is of the character m^{-2} but it remains to be investigated how the right side of (20b) behaves for very small values of mp because of the infinite value of the Bessel functions K of the second kind for $mp = 0$ and because the approximate representations of Nicholson for not too small values of the argument $mp = n!p$ have been used.

For small values of mp , the two Bessel functions of imaginary argument can be written

$$I_n(n\xi) = \frac{1}{n!} \left(\frac{n\xi}{2} \right)^n$$

$$K_n(n\xi) = \frac{(n-1)!}{2} \left(\frac{n\xi}{2} \right)^{-n}$$

The terms used for the power series for $\Gamma^* = \sum A_p \Gamma^*_p$ are:

$$\Gamma^*_p = \rho^{p+2} (\rho_a - \rho) \quad (23)$$

giving for the terms under the integrals in (20)

$$\frac{d}{d\xi} \left(\xi \frac{d\Gamma^*_p}{d\xi} \right) = \xi^{p+1} [(p+2)^2 \rho_a - (p+3)^2 \xi]$$

The first integral in (20), if it is to be computed only for a very small upper limit ρ , is then written:

$$\int_0^\rho \left(\frac{m\xi}{2} \right)^m \frac{1}{m!} \left[(\rho+2)^2 \rho_a \xi^{p+1} - (p+3)^2 \xi^{p+2} \right] d\xi$$

Evaluating this expression and multiplying by $K_m/2\pi\rho$, the result is:

$$\frac{K_m}{2\pi\rho} \int_0^\rho I_{md} \left(\xi \frac{d\Gamma^*_p}{d\xi} \right) = \frac{\rho^{p+1}}{4\pi m} \left[(p+2)^2 \frac{\rho_a}{p+2+m} - (p+3)^2 \frac{\rho}{p+3+m} \right] \quad (a)$$

This equation proves that the curve showing the induction of inflow-angle change i by the vortices inside of the point considered starts from zero at $\rho = 0$ with the angle tangent

$$\tan \varphi_0 = \frac{1}{4\pi m} \left[\rho_a \rho^p \frac{(p+1)(p+2)^2}{p+2+m} - \rho^{p+1} \frac{(p+2)(p+3)^2}{p+3+m} \right]$$

Hence, for all values of $p > 0$, the induction curve is tangent to the abscissa axis and, for $p = 0$,

$$(\tan \varphi_0)_{p=0} = \frac{\rho_a}{\pi m(m+2)}$$

The second integral of (20) must also be considered in order to judge the trend of the contribution to the angle function i . It may be repeated here

$$\frac{I_m}{2\pi\rho} \int_\rho^{\rho_a} K_{md} \left(\xi \frac{d\Gamma^*}{d\xi} \right)$$

Now the integral has been computed by means of the asymptotic approximations of Nicholson for all lower limits

down to $\rho_1 = 0.6$. Therefore only the additional contributions from ρ to ρ_1 are considered: namely,

$$\begin{aligned} \frac{I_m}{2\pi\rho} \int_{\rho}^{\rho_1} K_m d\left(\xi \frac{d\Gamma^*}{d\xi}\right) &= \\ &= \frac{\rho^{m-1}}{4\pi m} \left[(p+2)^2 \frac{\rho_1^{p+2-m} - \rho^{p+2-m}}{\rho + 2 - m} - (p+3)^2 \frac{\rho_1^{p+3-m} - \rho^{p+3-m}}{\rho + 3 - m} \right] \quad (b) \end{aligned}$$

Excluding at first the particular cases $p + 2 - m = 0$ or $p + 3 - m = 0$ and considering that the lowest value of m is 2, one sees that, in the limit $\rho \rightarrow 0$, the contribution is zero and therefore, as the total integral from ρ to ρ_1 is finite and has to be multiplied by the zero value of $I_m/2\pi\rho$, the induction curves for the influence of the outside vortices go to zero at $\rho = 0$ tangentially to the abscissa axis, for $m > 2$, except the particular cases.

The particular cases are:

for $p = 0$; $m = 2, m = 3$

for $p = 1$; $m = 4$

for $p = 2$; $m = 4, m = 5$

Calling the exponent that becomes zero q , the expression to be evaluated is:

$$\left(\frac{\rho_1^q - \rho^q}{q} \right)_{q=0} = \rho_1^q \log \rho_1 - \rho^q \log \rho = \log \frac{\rho_1}{\rho}$$

Now for the orders $p = 0$ and $p = 2$, one has to apply the preceding result to the first term inside the brackets of (b); whereas, for $p = 1$, the second term gives the particular case.

These results

$$p = 0, \quad n = 2$$

$$\frac{I_a}{2\pi\rho} \int_{\rho}^{\rho_1} K_{,d} \left(\xi \frac{d\Gamma_o^*}{d\xi} \right) = \frac{\rho}{8\pi} \left[4\rho_a \log \frac{\rho_1}{\rho} - 9(\rho_1 - \rho) \right] \quad (b_{0,2})$$

$$p = 2, \quad n = 4$$

$$= \frac{\rho^3}{16\pi} \left[16\rho_a \log \frac{\rho_1}{\rho} - 25(\rho_1 - \rho) \right] \quad (b_{2,4})$$

$$p = 1, \quad n = 4$$

$$= \frac{\rho^3}{16\pi} \left[-9(\rho_1 - \rho) - 16\rho_a \log \frac{\rho_1}{\rho} \right] \quad (b_{1,4})$$

These expressions show that, for the lower limit $\rho = 0$, the contributions are zero and therefore, as the contribution produced by the vortices at points with larger values of ρ computed with Nicholson's formulas are finite in the values proper of the integral and zero after multiplication by $I_a/2\pi\rho$, it appears that the induction of the outside vortices on the root point in the particular cases previously computed (for $\rho = 0$) is also zero.

Yet the influence of the logarithmic term appearing in $(b_{0,2})$, $(b_{2,4})$, and $(b_{1,4})$ causes the tangent of the induction curve to be perpendicular to the abscissa axis at $\rho = 0$.

DETERMINATION OF THE CONSTANT λ AND OF THE CIRCULATION FACTORS FROM THE TORQUE COEFFICIENT AND FROM THE INDUCTION OF MINIMUM LOSS

The torque coefficient has been expressed by the integral (5b). This integral shows that, to each series term of

$$\Gamma^* = \sum A_p \Gamma_p^*, \quad \Gamma_p^* = \rho^{p+2} (\rho_a - \rho) \quad (23)$$

there corresponds a contribution of the torque coefficient of minimum loss of energy:

$$C_{Q,p} = \int_0^{\rho_a} d\rho \rho^{p+3} (\rho_a - \rho) \left[1 + \lambda \frac{\rho^2}{1 + \rho^2} + \frac{\delta}{2} \rho \right] (1 - k\delta) \quad (5c)$$

The integration leads to (δ is assumed to be constant and in one place $\delta/2$ is neglected against unity):

$$\begin{aligned} \frac{C_{Q,p}}{1-k\delta} = & -\lambda \left[\frac{\rho_a^{p+1}}{p+1} + \rho_a^{p+3} \frac{1}{(p+2)(p+3)} - \rho_a^{p+5} \frac{1}{(p+4)(p+5)} \right] \\ & + \lambda \rho_a \int_0^{\rho_a} \frac{d\rho \rho^{p+1}}{1+\rho^2} + \lambda \int_0^{\rho_a} \frac{d\rho \rho^p}{1+\rho^2} \\ & + \rho_a^{p+5} \frac{1}{(p+4)(p+5)} + \rho_a^{p+6} \frac{\delta}{2(p+5)(p+6)} \quad (5d) \end{aligned}$$

As the terms for Γ^*_p have been used only for the numbers, $p = 0$, $p = 1$, and $p = 2$, there are also only three contributory terms for C_Q : namely,

$$\begin{aligned} \frac{C_{Q,0}}{1-k\delta} = & -\lambda \left[\rho_a + \frac{\rho_a^3}{6} - \frac{\rho_a^5}{20} - \arctan \rho_a - \frac{\rho_a}{2} \log(1 + \rho_a^2) \right] \\ & + \frac{\rho_a^5}{20} + \delta \frac{\rho_a^6}{60} \end{aligned}$$

$$\begin{aligned} \frac{C_{Q,1}}{1-k\delta} = & -\lambda \left[-\frac{\rho_a^2}{2} + \frac{\rho_a^4}{12} - \frac{\rho_a^6}{30} + \rho_a \arctan \rho_a - \frac{1}{2} \log(1 + \rho_a^2) \right] \\ & + \frac{\rho_a^6}{30} + \delta \frac{\rho_a^7}{84} \end{aligned}$$

$$\begin{aligned} \frac{C_{Q,2}}{1-k\delta} = & -\lambda \left[-\frac{\rho_a^3}{6} + \frac{\rho_a^5}{20} - \frac{\rho_a^7}{42} + \frac{\rho_a}{2} \log(1 + \rho_a^2) - \rho_a + \arctan \rho_a \right] \\ & + \frac{\rho_a^7}{42} + \delta \frac{\rho_a^8}{112} \end{aligned}$$

The total torque coefficient is then to be written as follows:

$$C_Q = A_0 C_{Q,0} + A_1 C_{Q,1} + A_2 C_{Q,2} \quad (50)$$

As not only the $C_{Q,p}$ terms but also the factors A_p , according to (24a) are linear functions of λ , there will finally arise a quadratic equation for λ .

The equations determining the factors A_p have been stated in (24a). They may be treated now somewhat more explicitly in such a way as to show how the curves computed to give the induction of inflow angle i_p by the corresponding circulation term Γ_p^* are to be used to find the coefficients of the equations (24a) determining the factors A_p .

Writing then equation (24a) in the form:

$$\left. \begin{aligned} A_0 f_{00} + A_1 f_{01} + A_2 f_{02} &= \lambda f_0 - \frac{\delta}{2} \epsilon_0 \\ A_0 f_{10} + A_1 f_{11} + A_2 f_{12} &= \lambda f_1 - \frac{\delta}{2} \epsilon_1 \\ A_0 f_{20} + A_1 f_{21} + A_2 f_{22} &= \lambda f_2 - \frac{\delta}{2} \epsilon_2 \end{aligned} \right\} \quad (24b)$$

the factors f and g have the following significance:

$$\left. \begin{aligned} f_{np} &= \int_{\rho_1}^{\rho_a} d\rho \rho \, i_n \, i_p \\ f_n &= \int_{\rho_1}^{\rho_a} \frac{d\rho \rho^2}{1+\rho^2} i_n \\ \epsilon_n &= \int_{\rho_1}^{\rho_a} d\rho \rho \, i_n \end{aligned} \right\} \quad (27)$$

which shows that

$$f_{np} = f_{pn}$$

NUMERICAL RELATION BETWEEN THE FACTOR λ OF IDEAL
ANGLE CHANGE FUNCTION i AND THE PRESCRIBED
TORQUE (POWER) COEFFICIENT

As a first case has been treated the propeller of tip
radius number $\rho_a = \sqrt{24} = 4.9$. Formula (5d) gives

$$C_{Q,0} = (1 - k\delta)(141.0 + 125.9\lambda + 230.4\delta)$$

$$C_{Q,1} = (1 - k\delta)(461.0 + 419.9\lambda + 806.0\delta)$$

$$C_{Q,2} = (1 - k\delta)(1,612.0 + 1,489.7\lambda + 2,982.0\delta)$$

Supposing now that equations (24b) for A_0, A_1, A_2 have
been solved in the form:

$$A_0 = \alpha_0\lambda + \beta_0\delta$$

$$A_1 = \alpha_1\lambda + \beta_1\delta$$

$$A_2 = \alpha_2\lambda + \beta_2\delta$$

and that the value of the torque (or power) coefficient
 C_Q is prescribed by the performance of the engine and the
airplane, then the following quadratic equation for λ is
obtained:

$$C_Q = (1 - k\delta) [(\alpha_0\lambda + \beta_0\delta)(141.0 + 125.9\lambda + 230.4\delta) \\ + (\alpha_1\lambda + \beta_1\delta)(461.0 + 419.9\lambda + 806.0\delta) \\ + (\alpha_2\lambda + \beta_2\delta)(1,612.0 + 1,489.7\lambda + 2,982.0\delta)]$$

or

$$\lambda^2(125.9\alpha_0 + 419.9\alpha_1 + 1,489.7\alpha_2) + \lambda[(141.0 + 230.4\delta)\alpha_0 \\ + (461.0 + 806.0\delta)\alpha_1 + (1,612.0 + 2,982.0\delta)\alpha_2 + 125.9\delta\beta_0 \\ + 419.9\delta\beta_1 + 1,489.7\delta\beta_2] + \beta_0\delta(141.0 + 230.4\delta) \\ + \beta_1\delta(461.0 + 806.0\delta) + \beta_2\delta(1,612.0 + 2,982.0\delta) \\ = C_Q/(1 - k\delta)$$

Putting for a practical example $\delta = 0.02$, this equation takes the form

$$\begin{aligned} \lambda^2(125.9\alpha_0 + 419.9\alpha_1 + 1,489.7\alpha_2) + \lambda(145.6\alpha_0 + 477.1\alpha_1 \\ + 1,671.6\alpha_2 + 2.52\beta_0 + 8.4\beta_1 + 29.79\beta_2) + \beta_0 2.91 + \beta_1 9.51 \\ + \beta_2 33.43 - \frac{C_Q}{1 - k\delta} = 0 \end{aligned} \quad (26)$$

NUMERICAL DETERMINATION OF THE FACTORS A_p OF

CIRCULATION Γ_p^* AND ANGLE CHANGE i_p

Case of $\rho_a = 4.9$

In equation (24b), the factors f_{npp_a} , f_{np_a} , and g_{n,ρ_a} are definite integrals that must be determined from the angle-change curves i_{n,ρ_a} , which had been computed as the sums of the self-induction, the outside and the inside induction of the circulations Γ_{n,ρ_a}^* at each point of the radial abscissa ρ and which are shown in figures 17 to 24. The integrands of the integrals f and g of formulas (27) have been computed and plotted as figures 25, 26, and 27. From these curves, the areas have been taken.

Taking these values, the equations that have to determine the factors A_p are, for the radius number $\rho_a = 4.9$, numerically given by

$$A_0 \ 0.1067 + A_1 \ 0.347 + A_2 \ 1.236 = \frac{\lambda}{\rho_a} \ 0.3 - \frac{\delta}{2} \ 0.2153$$

$$A_0 \ 0.347 + A_1 \ 1.331 + A_2 \ 5.269 = \frac{\lambda}{\rho_a} \ 0.859 - \frac{\delta}{2} \ 0.696$$

$$A_0 \ 1.236 + A_1 \ 5.269 + A_2 \ 22.42 = \frac{\lambda}{\rho_a} \ 2.977 - \frac{\delta}{2} \ 2.457$$

The solution, which had to be computed very exactly because it is rather sensitive to small changes in the coefficients, was found to be:

$$\left. \begin{aligned} A_0 &= 9.4556 \frac{\lambda}{\rho_a} - 1.72385 \frac{\delta}{2} \\ A_1 &= -4.0469 \frac{\lambda}{\rho_a} - 0.23539 \frac{\delta}{2} \\ A_2 &= 0.56259 \frac{\lambda}{\rho_a} + 0.040708 \frac{\delta}{2} \end{aligned} \right\} \quad (24c)$$

so that:

$$\begin{aligned} \alpha_0 &= 1.930 & \beta_0 &= -0.8619 \\ \alpha_1 &= -0.8250 & \beta_1 &= -0.1177 \\ \alpha_2 &= 0.115 & \beta_2 &= 0.020354 \end{aligned}$$

The quadratic equation (26) determining the factor λ from the torque coefficient C_Q is then to be written as follows:

$$\lambda^2 68.0 + \lambda 76.6 - 2.95 - \frac{C_Q}{1 - k\delta} = 0 \quad (26a)$$

To choose a value for the (dimensionless) torque (or power) coefficient C_Q appropriate for the tip radius number (circumferential path per unit of advance) $\rho_a = 4.9$, one has to remember that so high a value corresponds to rather low power, low speed, and high revolution values; for instance

$$\begin{aligned} P &= 160 \text{ hp.} = 160.75 \text{ kg-m/sec} \\ n &= 1,190 \text{ r.p.m.}, \quad \omega = \frac{n\pi}{30} = 200 \text{ sec.}^{-1} \\ h &= 18,000 \text{ ft.}, \quad \mu = \frac{1}{2} \mu_0 = \frac{1}{16} \text{ kg-m-sec} \\ w &= 123.5 \text{ m.p.h.} = 55 \text{ r/sec} \\ d &= 8.54 \text{ ft.} \end{aligned}$$

$$C_Q = \frac{160 \times 75 \times 200^2}{1/16 \times 55^5} = 15.3$$

and, with $k = 7.5$ and $\delta = 0.02$,

$$\frac{C_Q}{1 - k\delta} = \frac{C_Q}{0.85} = 18.0$$

Equation (26a) becomes

$$\lambda^2 + \lambda 0.113 - 0.300 = 0$$

As only the positive root has a physical meaning, one obtains

$$\lambda = 0.2225$$

It must be remembered that this value of λ is valid if the ideally best distribution of induced angle changes is obtained. If the best distribution is only approximately obtained, the value of λ may change somewhat from the preceding value.

The ideal distribution of inflow angle change was stated to be

$$i_{id} = \lambda \frac{\rho}{1 + \rho^2} - \frac{\delta}{2}$$

It must now be shown how closely the ideal induced angle change function i_{id} has been approximated with only the three terms corresponding to $p = 0, 1$, and 2 by the method of the least average absolute error of equations (24a), (24b), and (27).

In the first column of table I are given the computed ideal induced angles and, in the next column, the angles i induced by the circulation Γ^*

$$\Gamma^* = A_0 \Gamma^*_0 + A_1 \Gamma^*_1 + A_2 \Gamma^*_2$$

taken from (23) and multiplied by the factors A_0 , A_1 , and A_2 , respectively.

The factors A_0 , A_1 , and A_2 can now be computed explicitly after the values of λ and δ are given from (24c), as follows:

$$A_0 = 1.93 \times 0.2225 - 0.0172 = 0.413$$

$$A_1 = -0.825 \times 0.2225 - 0.0023 = -0.1858$$

$$A_2 = 0.115 \times 0.2225 - 0.00041 = 0.0259$$

Table I

ρ/ρ_a	$A_0 i_0$	$A_1 i_1$	$A_2 i_2$	$\Sigma A_p i_p$	$\Sigma A_p \Gamma^*_p$
0.1	-	0.109	-0.02412	-	0.350
.15	-0.036	-	-	-	-
.2	.01509	.1043	-.04605	0.07324	.964
.3	.1102	.0382	-.0427	.1057	1.970
.4	.1818	-.082	-.0184	.0814	1.85
.5	.2265	-.1974	.0373	.0664	1.67
.6	.2490	-.302	.109	.0560	1.56
.7	.2423	-.379	.184	.0473	1.41
.8	.2135	-.418	.2466	.0421	1.28
.9	.154	-.366	.248	.036	.92
1.0	.053	-.141	.118	.031	0

Figure 28 shows the two curves of inflow angle change plotted to the same scale. The ideal curve for minimum energy loss agrees quite satisfactorily with the curve obtained from the integration over the three circulation (or vortex distribution) terms Γ^*_0 , Γ^*_1 , and Γ^*_2 down to the third of the radius; farther inside toward the hub, the curve found from the circulation power series, although it shows the same behavior in respect to a peak and the sudden decrease, needs some shifting over to the center. This change could be effected by a fourth term of the power series in Γ^* but, as the higher terms shift the circulation and also the inflow angle change more to the tip, it might be more satisfactory to add the lower term Γ^*_{-1} which had been avoided before

$$\Gamma^*_{-1} = \rho(\rho_a - \rho)$$

that is, the simple parabola. This term was not taken in order to prevent singularity at the center but, as an in-

ner circle of the hub has to be cut out anyhow, as has been explained in an earlier section, this singularity would not appear and would have no effect. This result proves that, with very few terms of the power series for the circulation, that is, for the thrust distribution, the ideal inflow deviation of minimum loss of energy can be approximated very closely.

In table I and figure 27 are given and plotted the ordinates for the resultant circulation.

$$\Gamma^* = A_0 \Gamma^*_0 + A_1 \Gamma^*_1 + A_2 \Gamma^*_2$$

The curve shows a peculiar sudden decrease toward the root. To the ideal inflow angle change

$$i_{id} = \lambda \frac{\rho}{1 + \rho^2} - \frac{\delta}{2}$$

corresponds an (ideal) circulation function Γ^*_{id} , which can be approximated the more closely the more terms Γ^*_p are used. Still a sufficiently close approximation will always be possible with a small number of appropriate series terms.

The other curves corresponding to figure 28 and to the other tip radii numbers $\rho_n = \sqrt{3}, \sqrt{8},$ and $\sqrt{15}$ will be determined in the same way.

Armour Institute of Technology,
Chicago, Ill., October 17, 1939.

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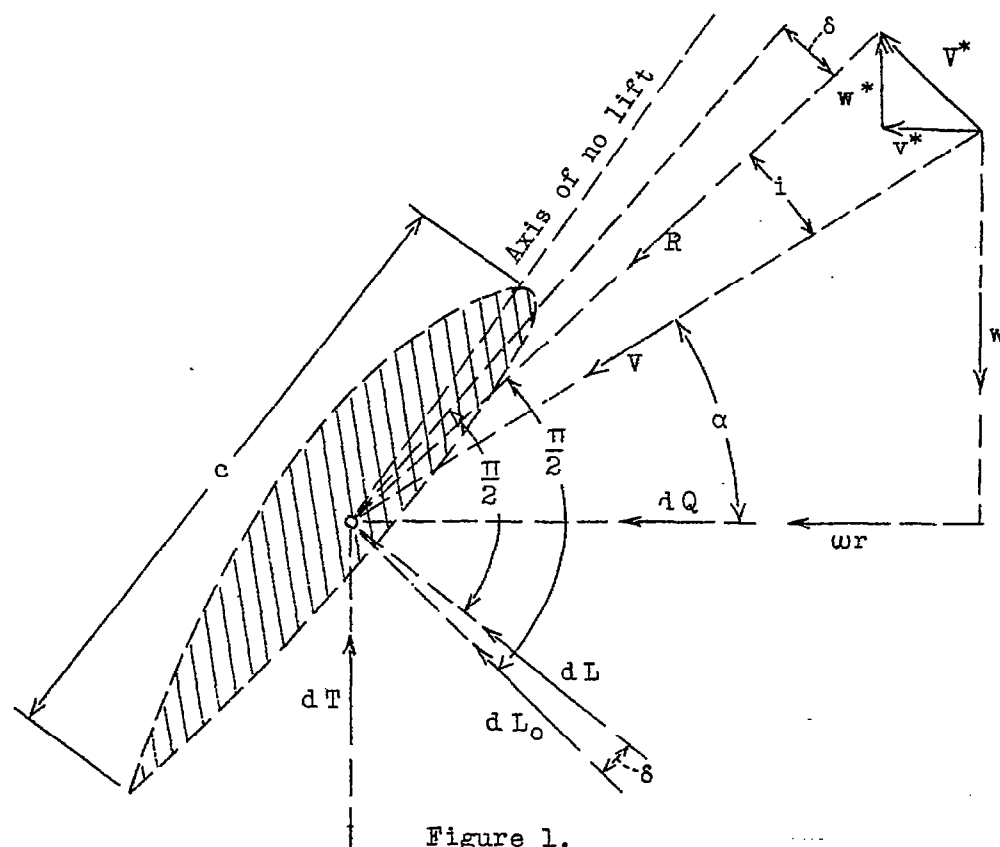


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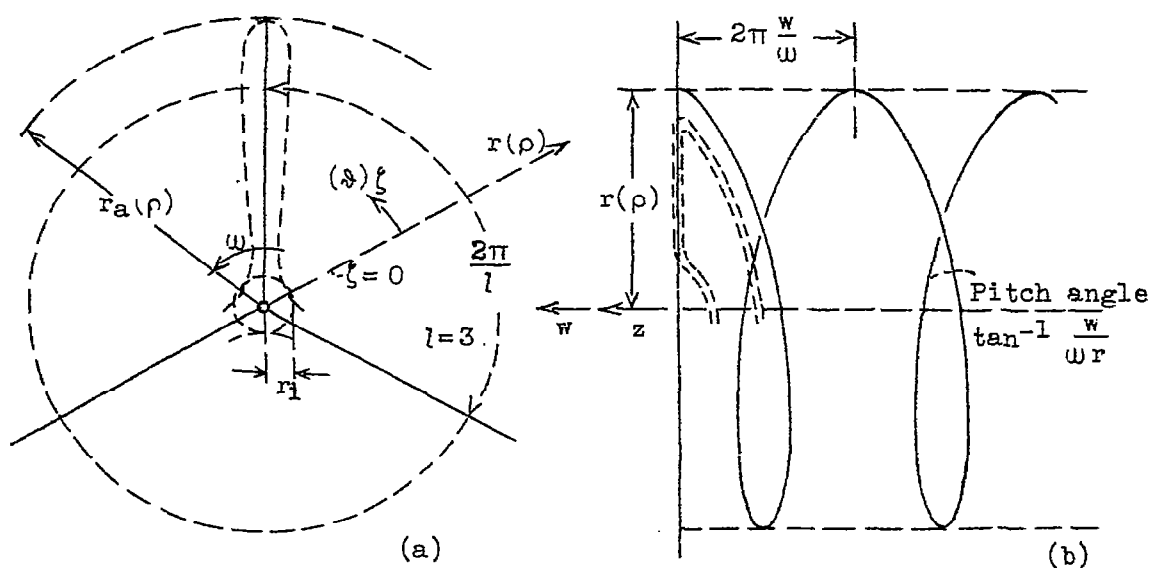


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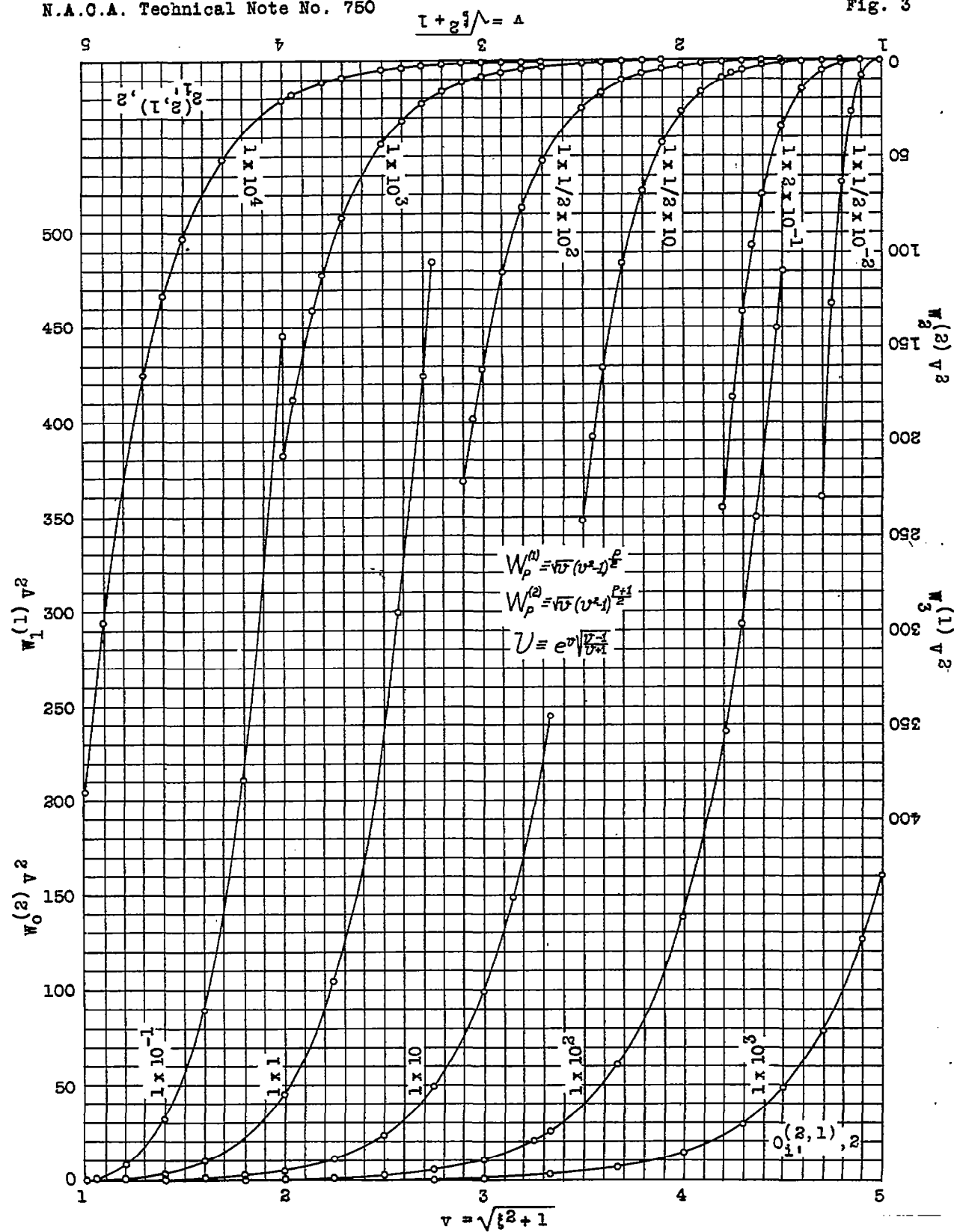


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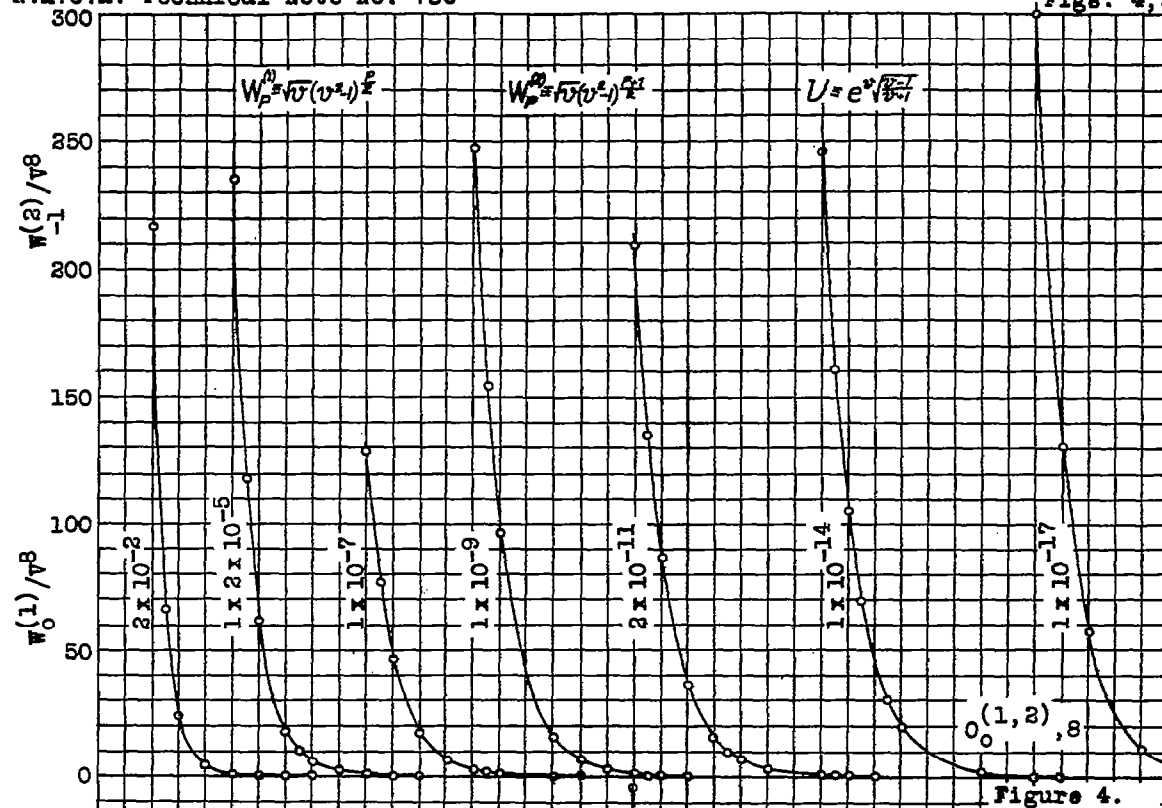


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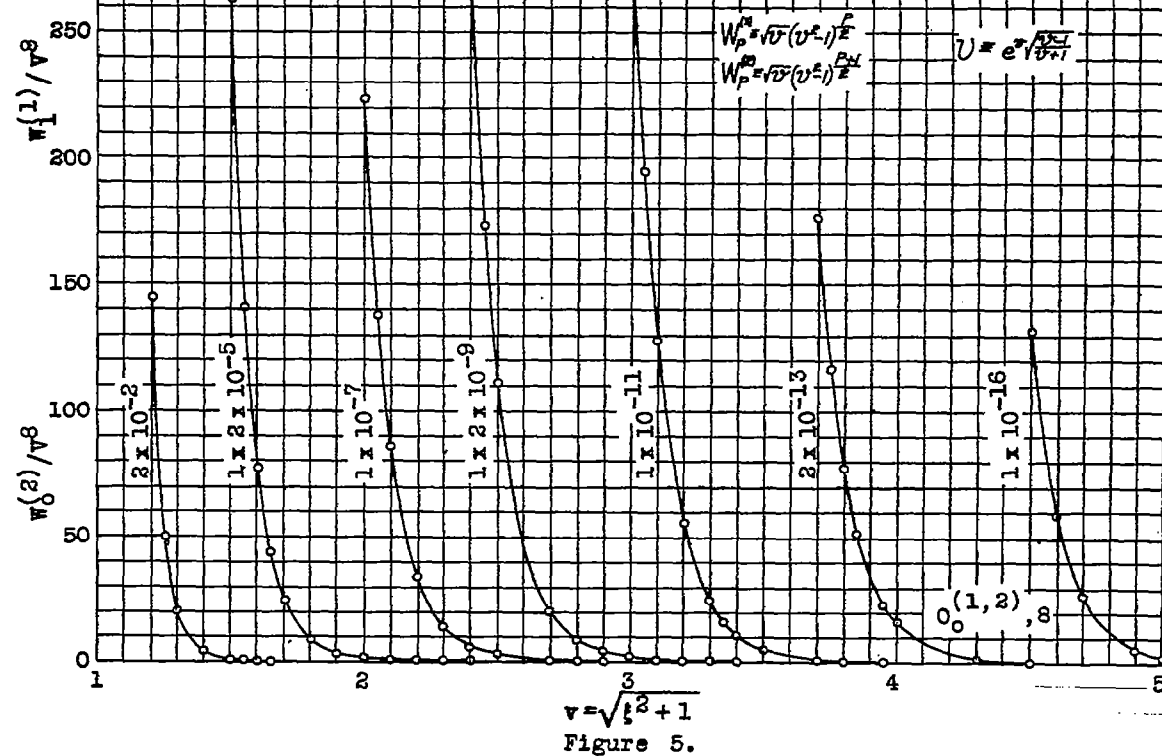


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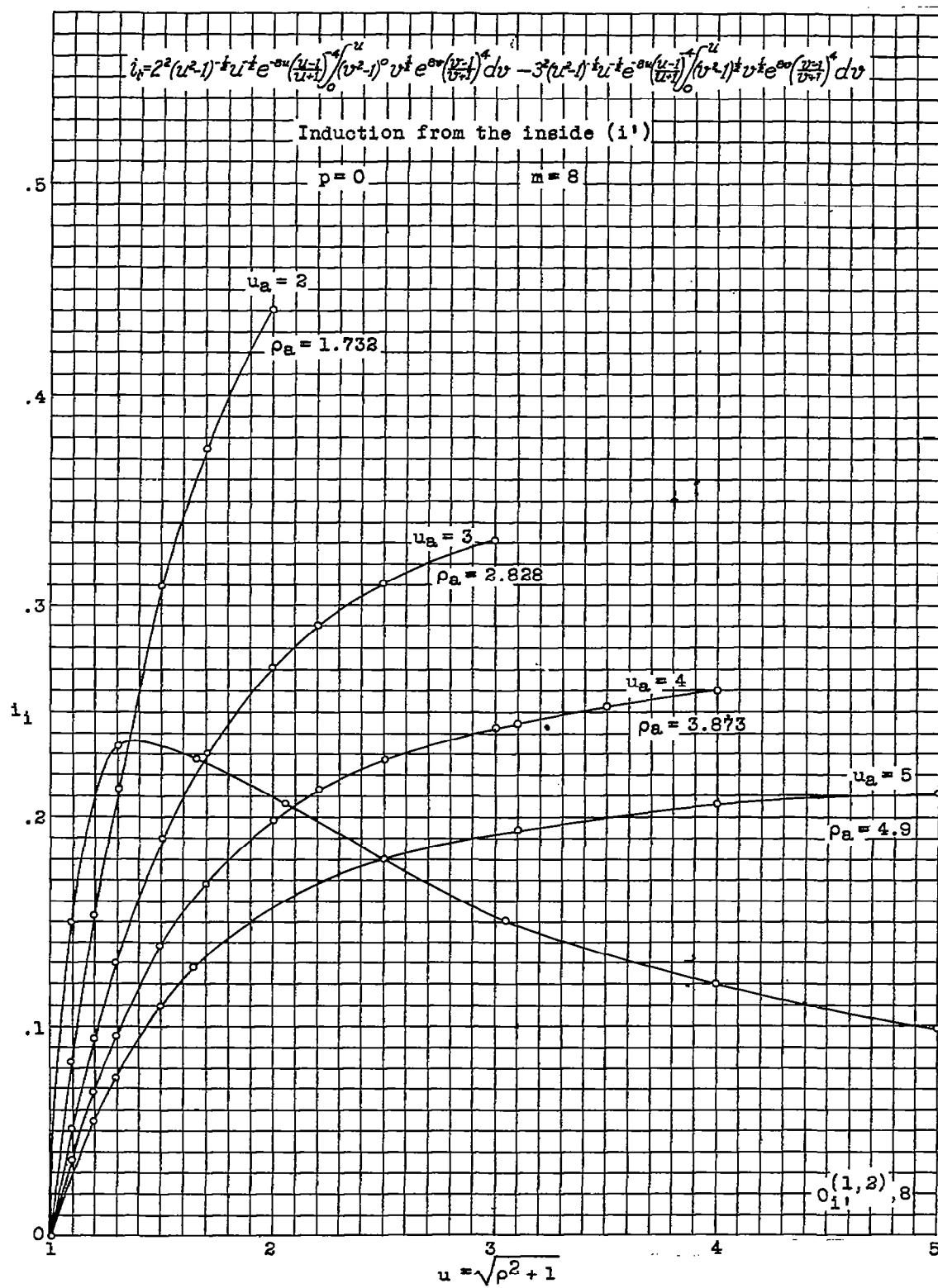
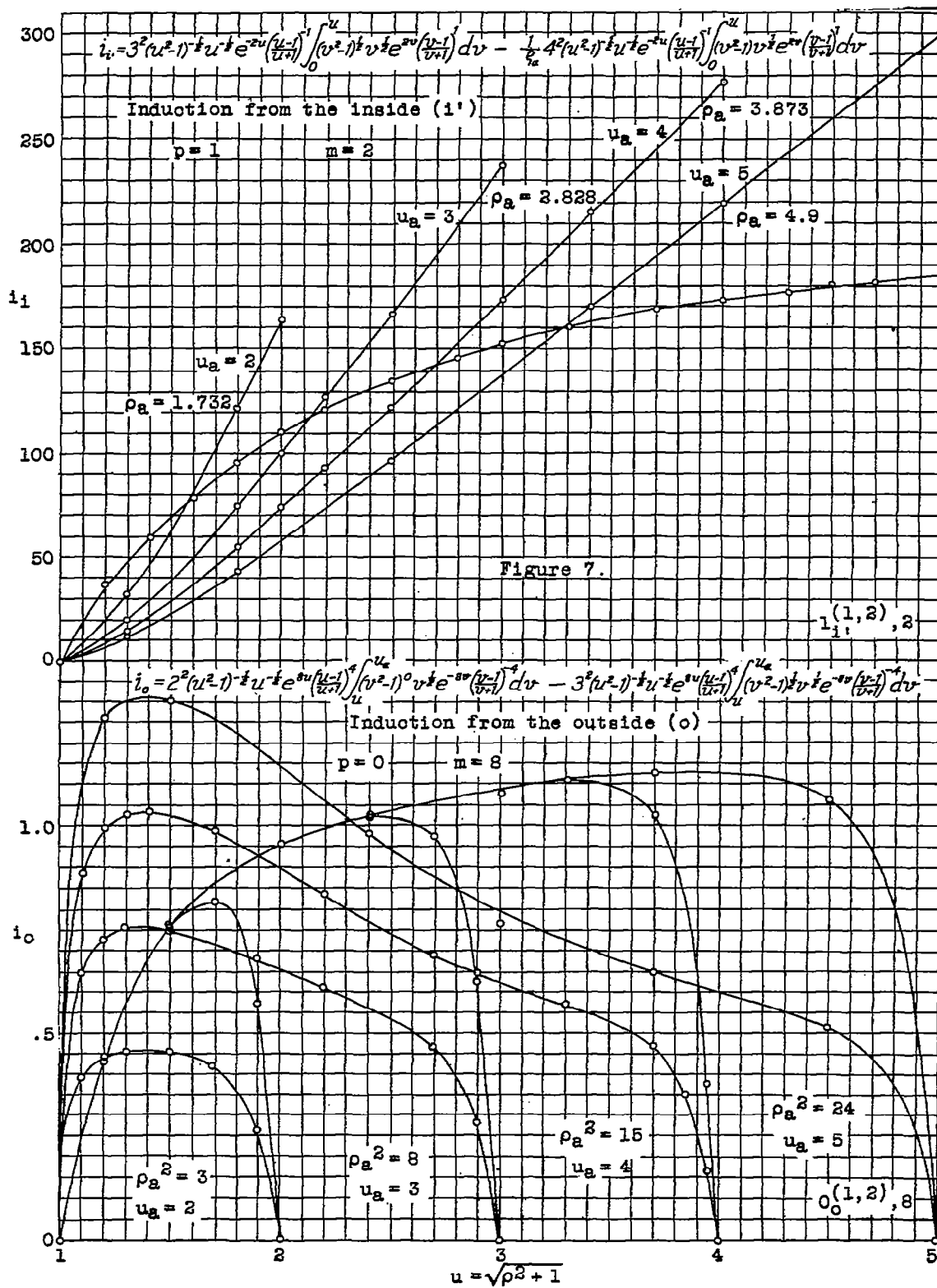
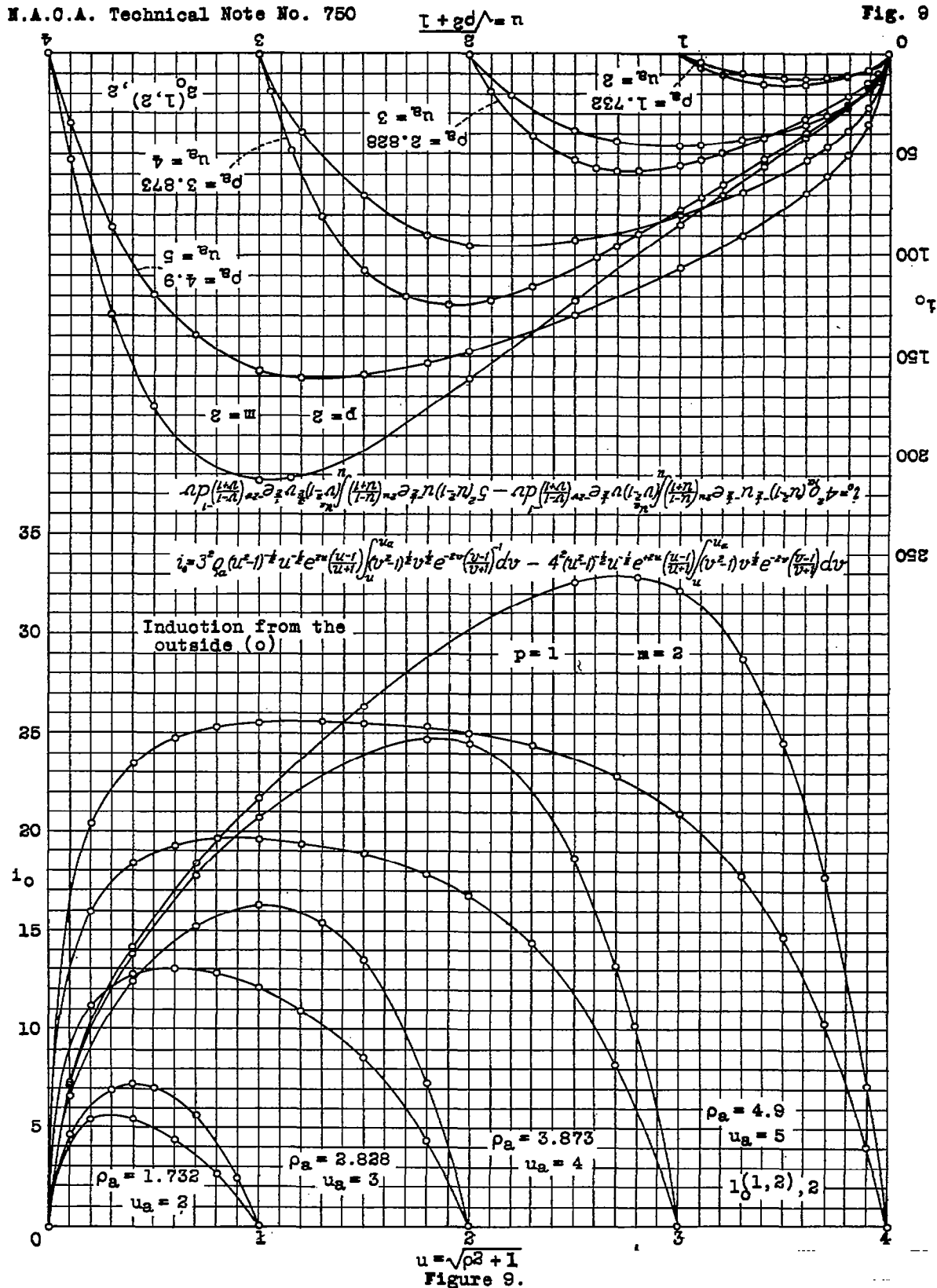


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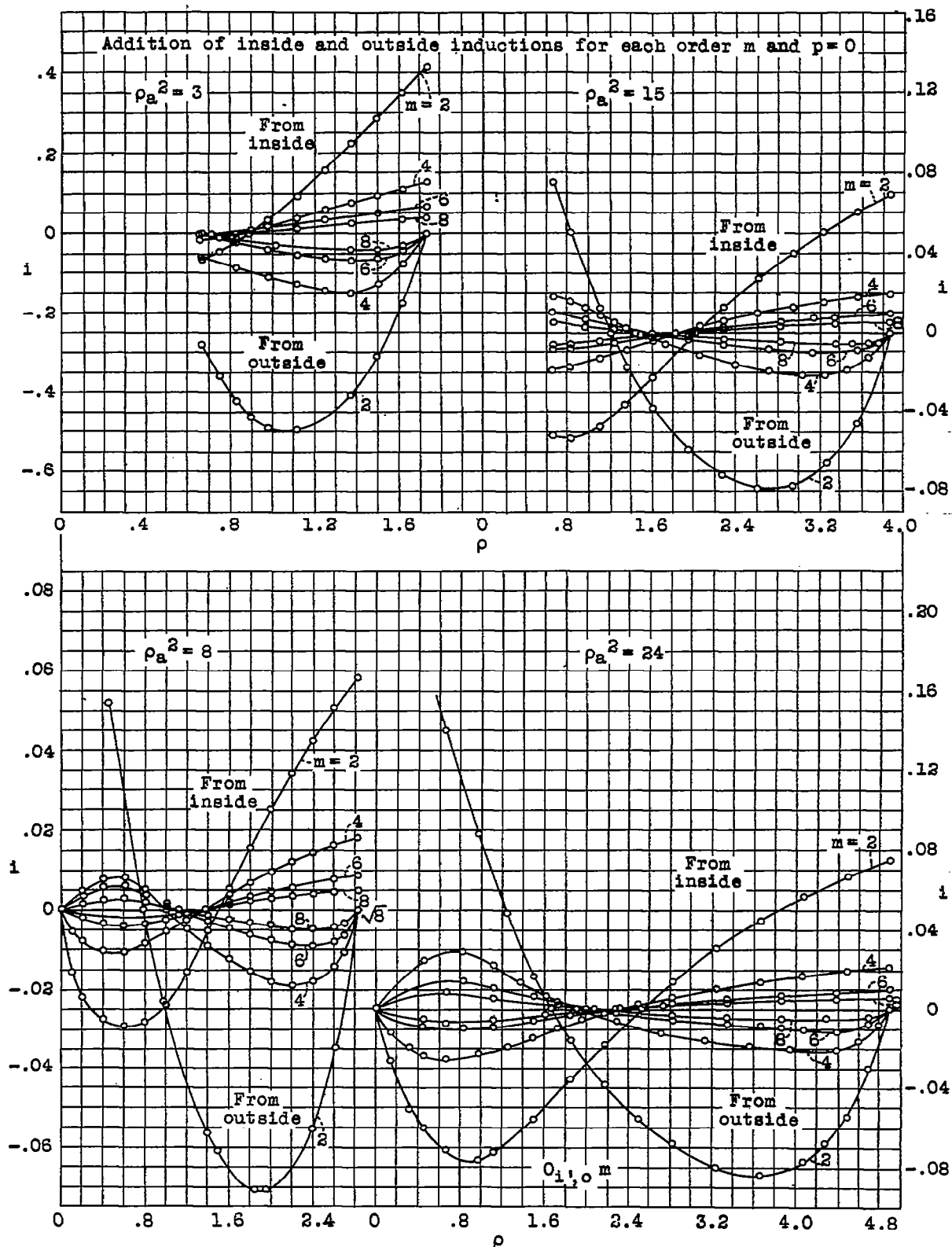


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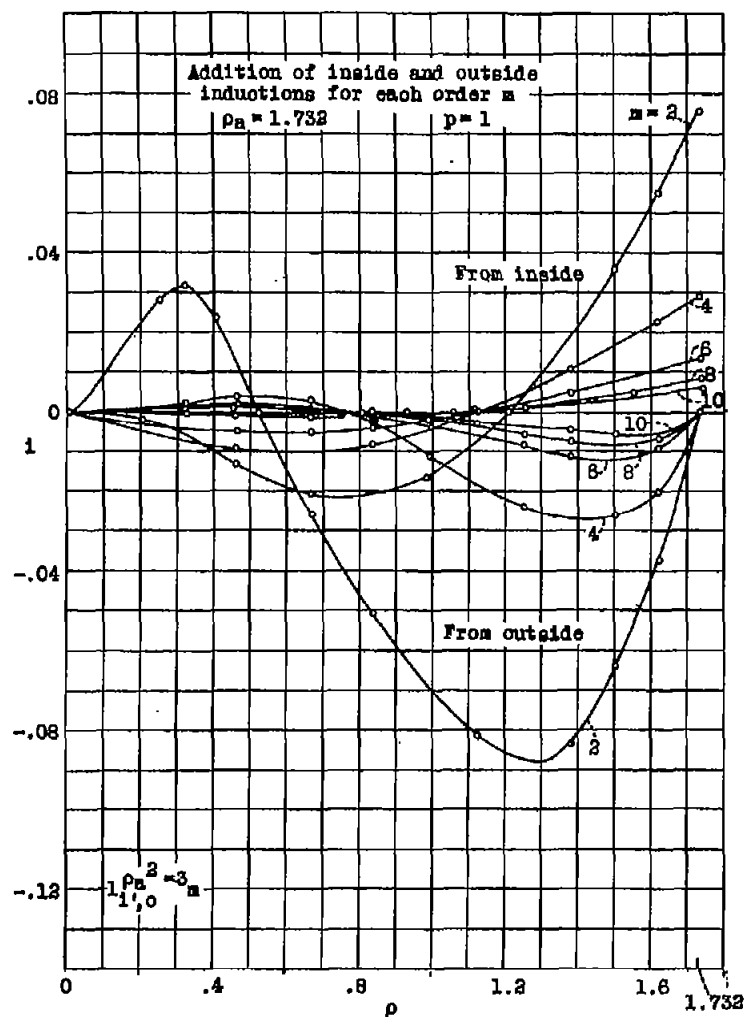


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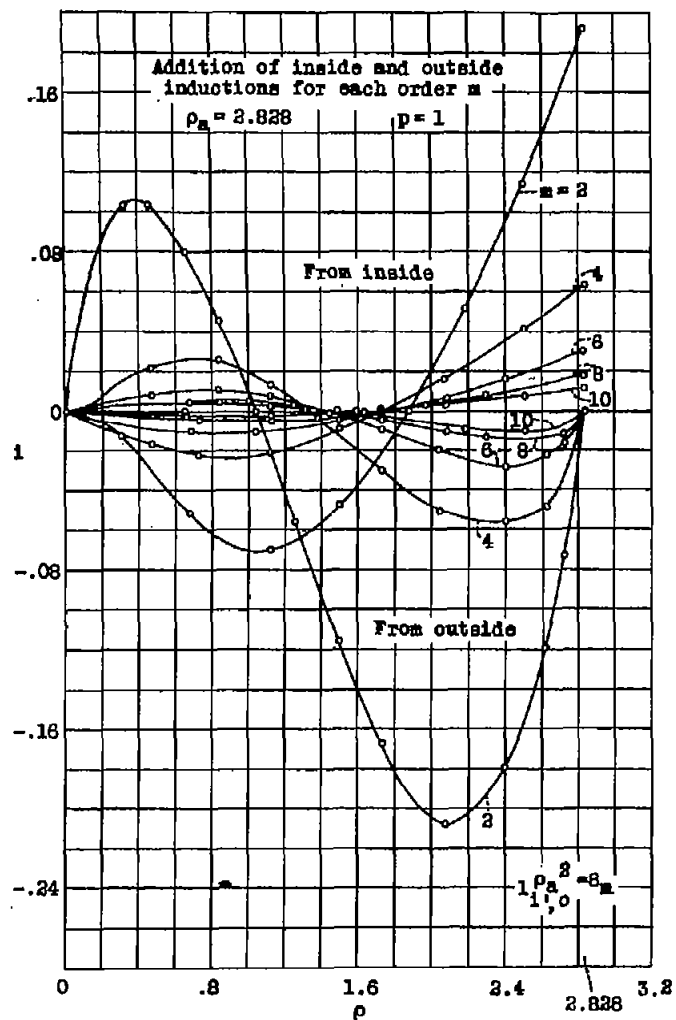


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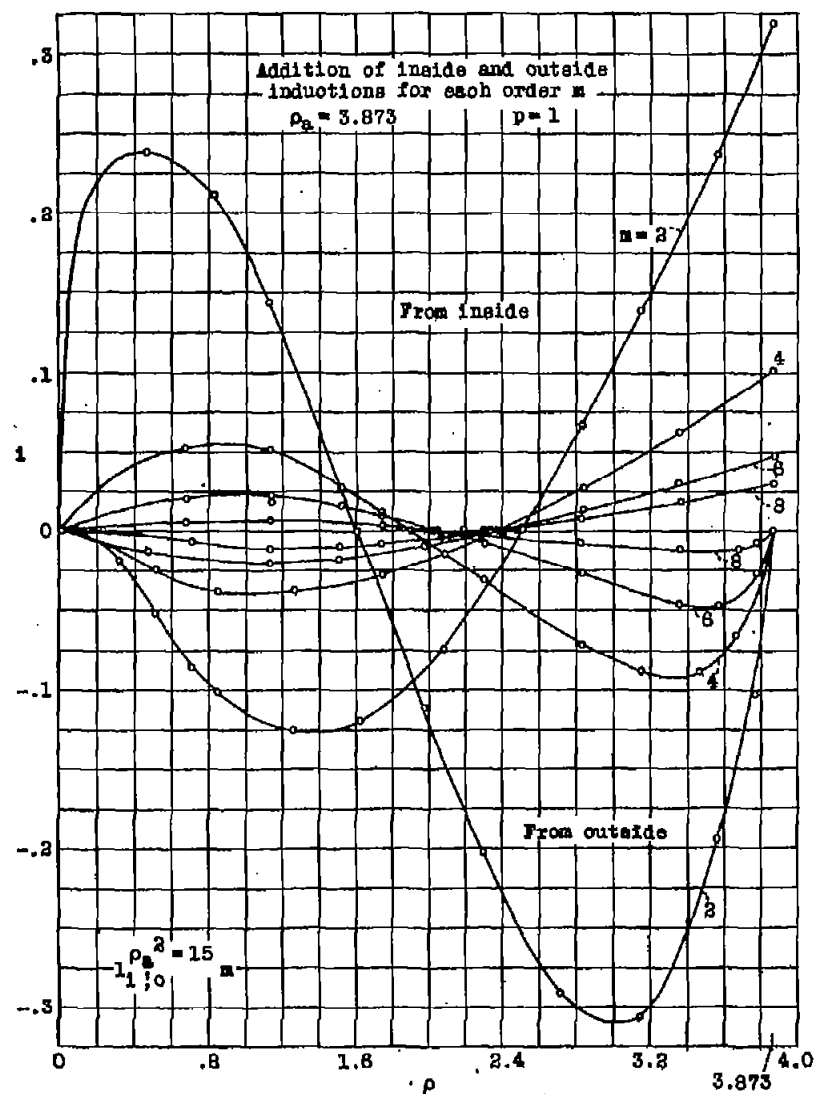


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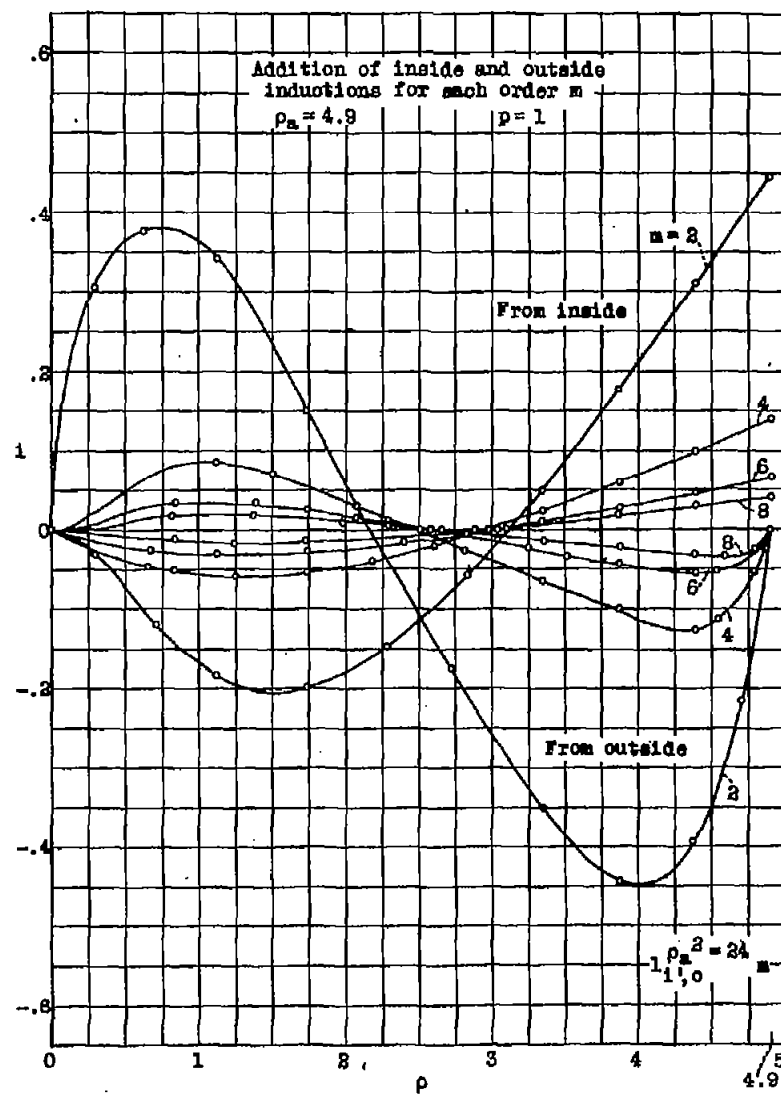


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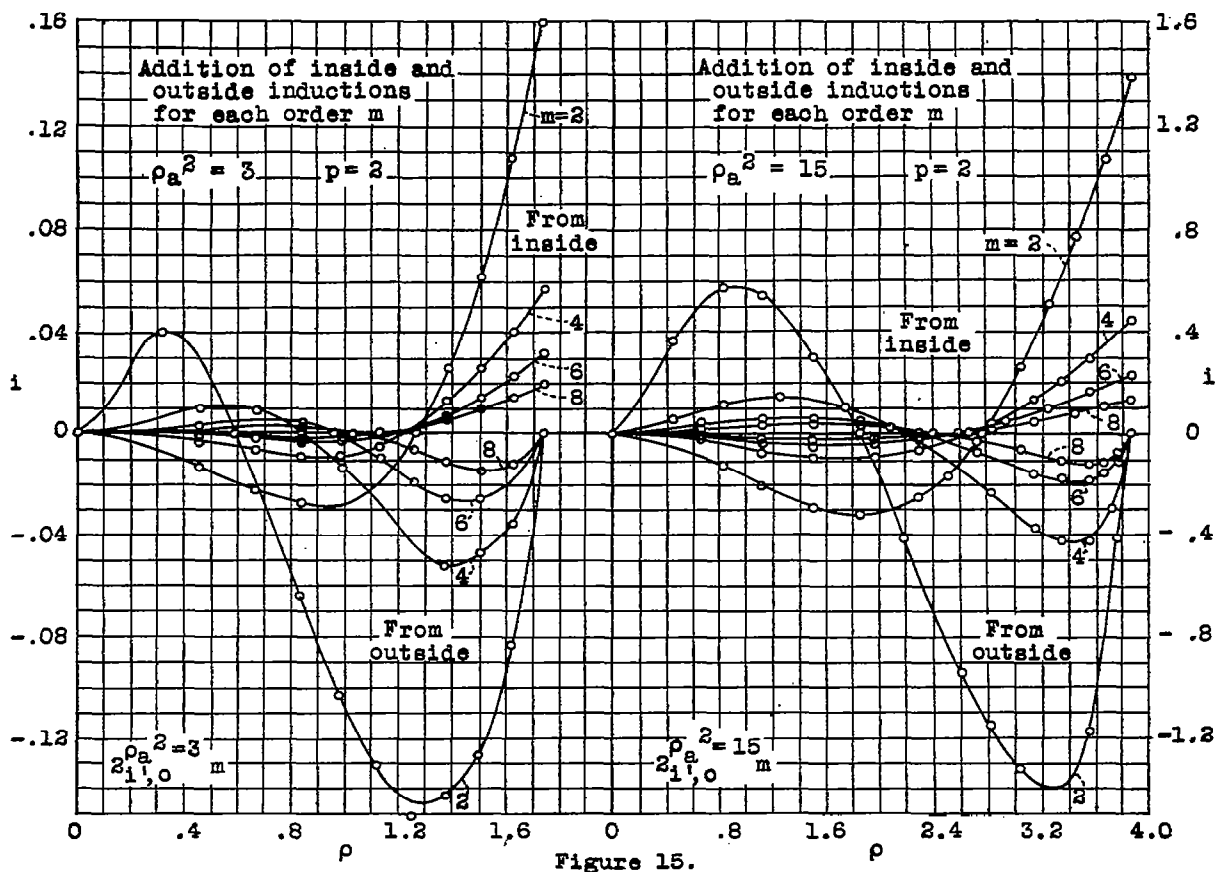


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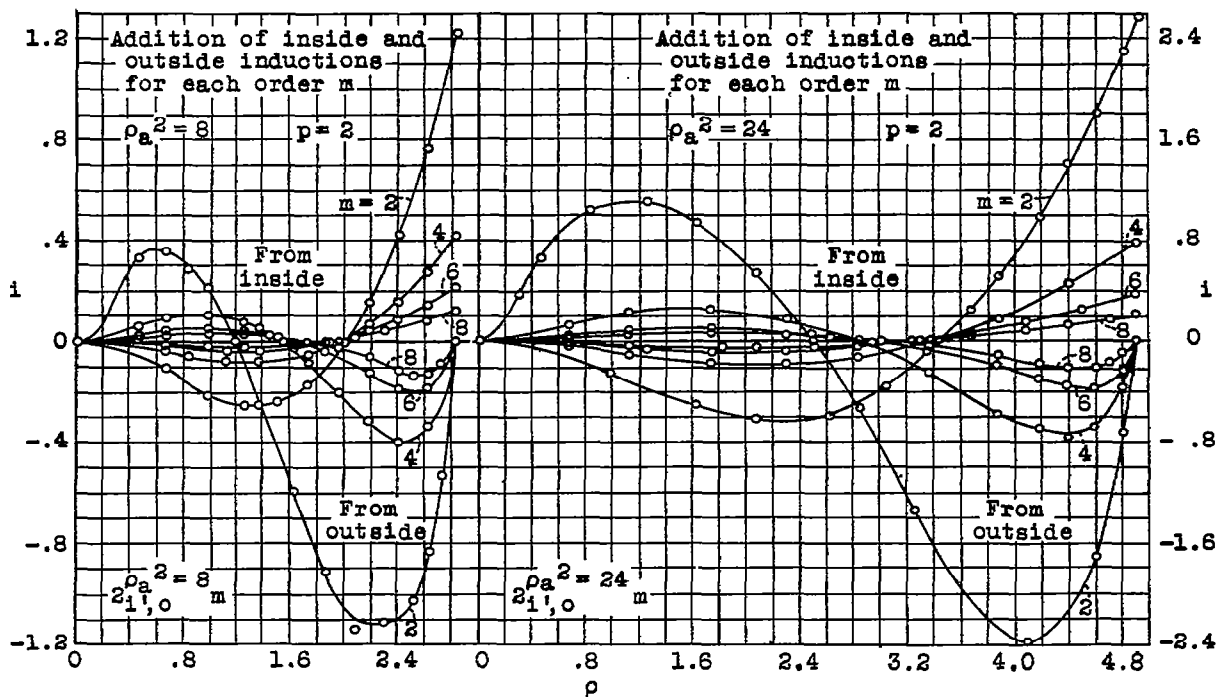


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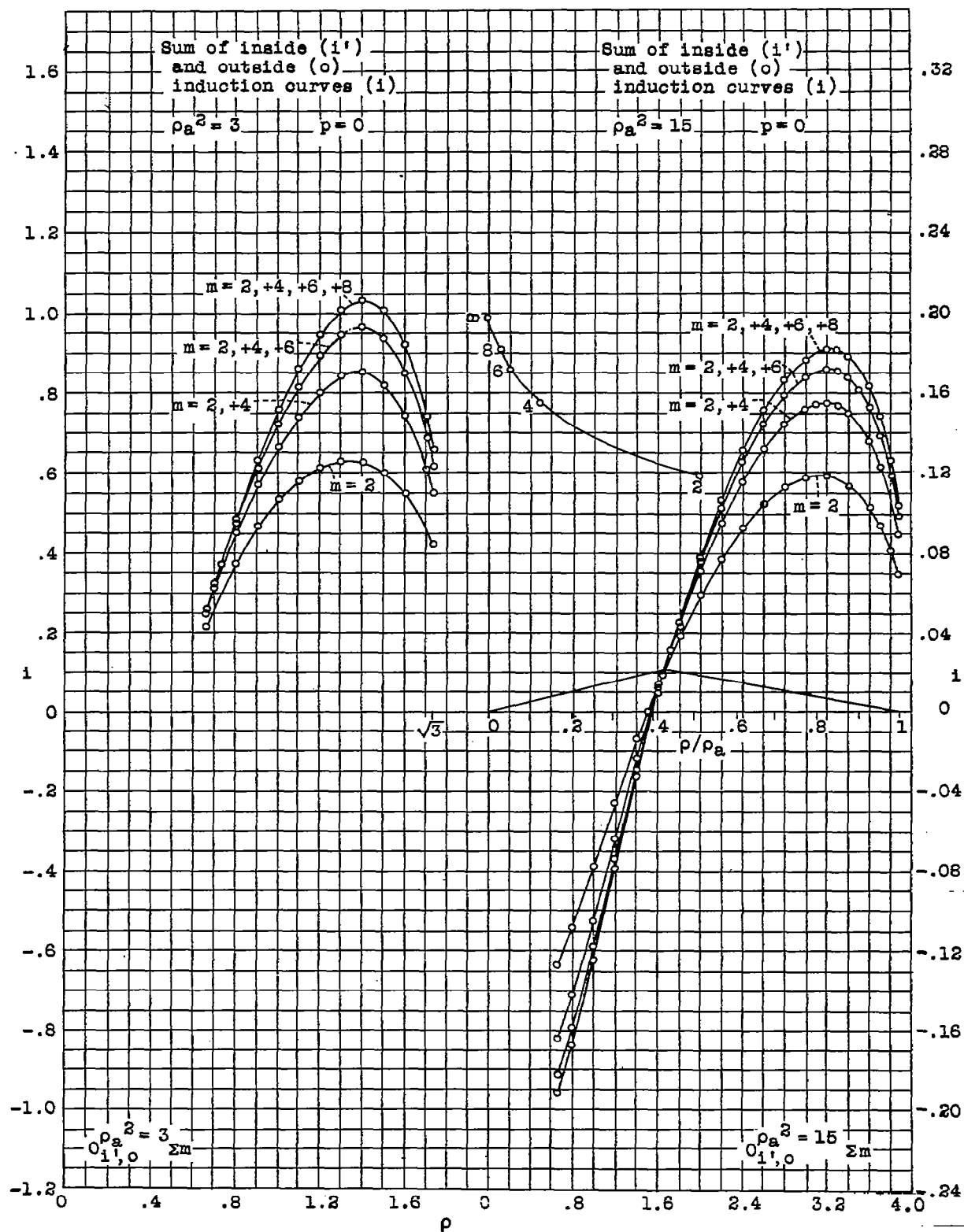


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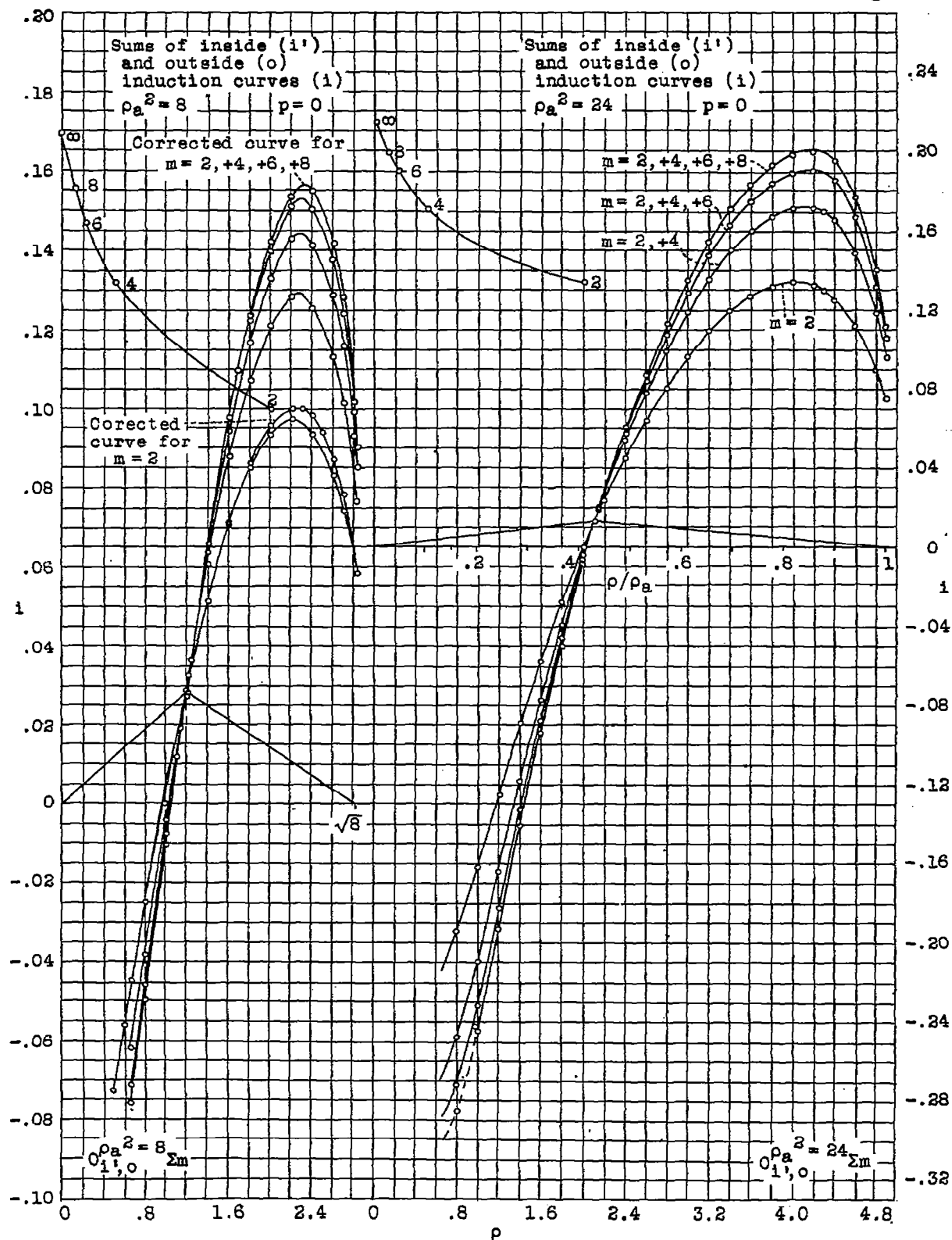
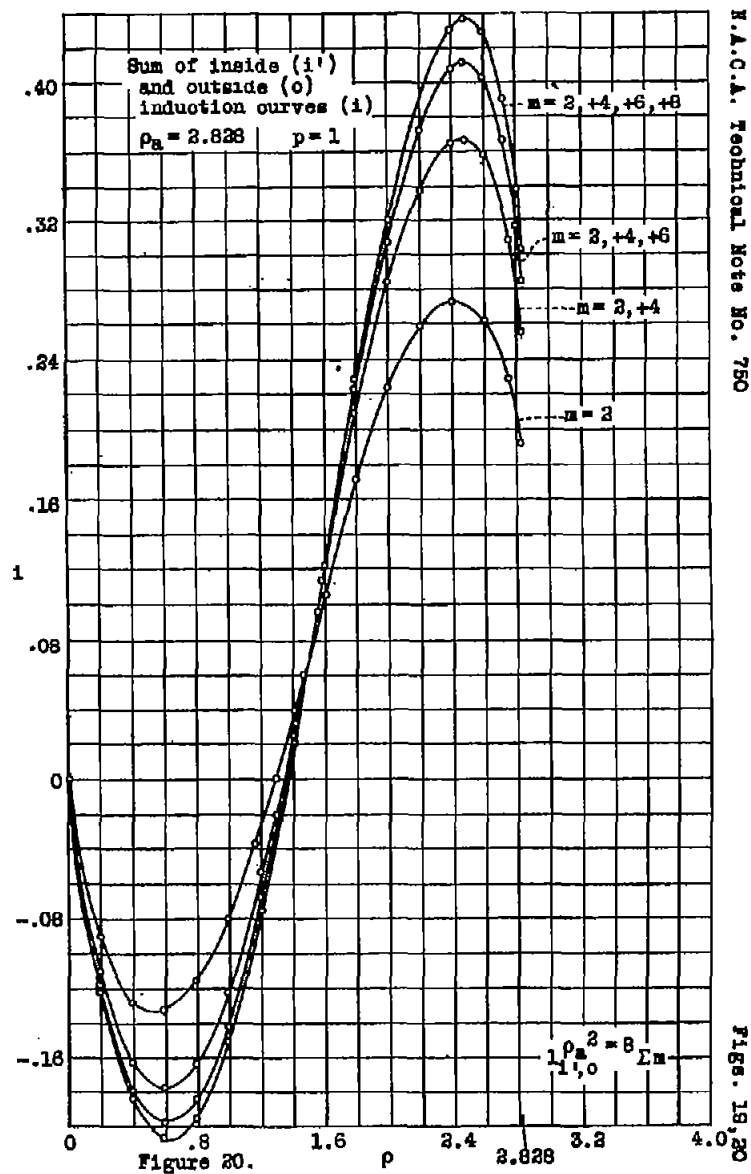
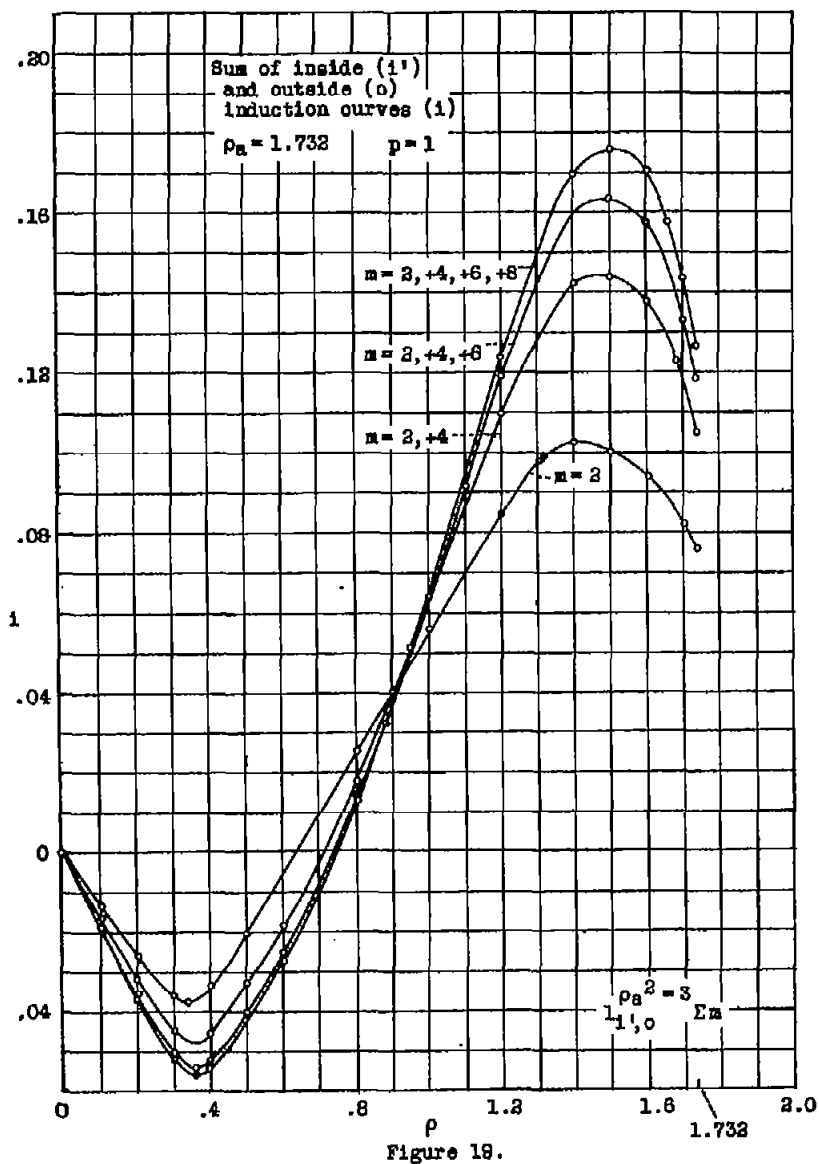


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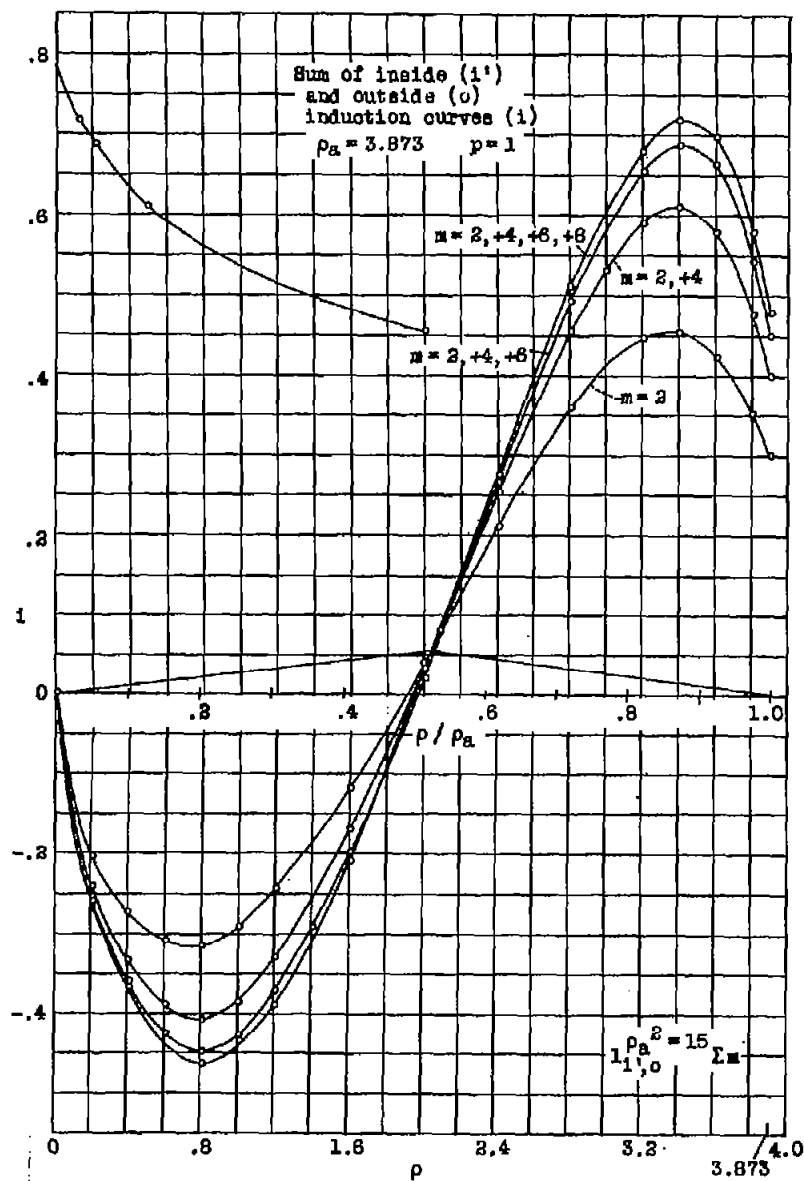


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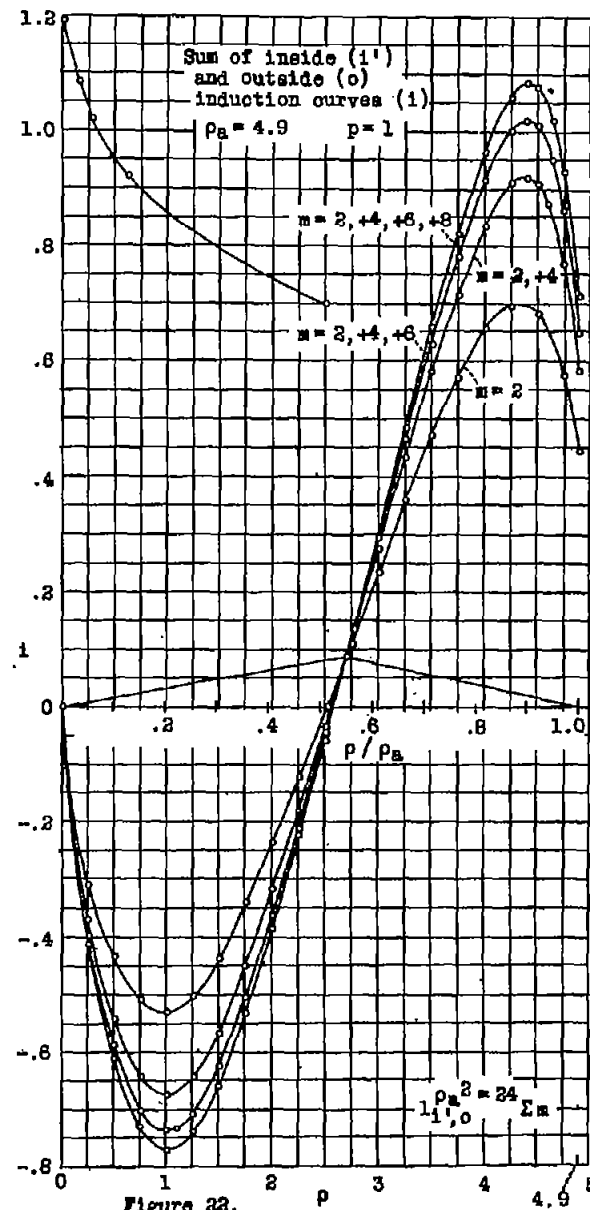


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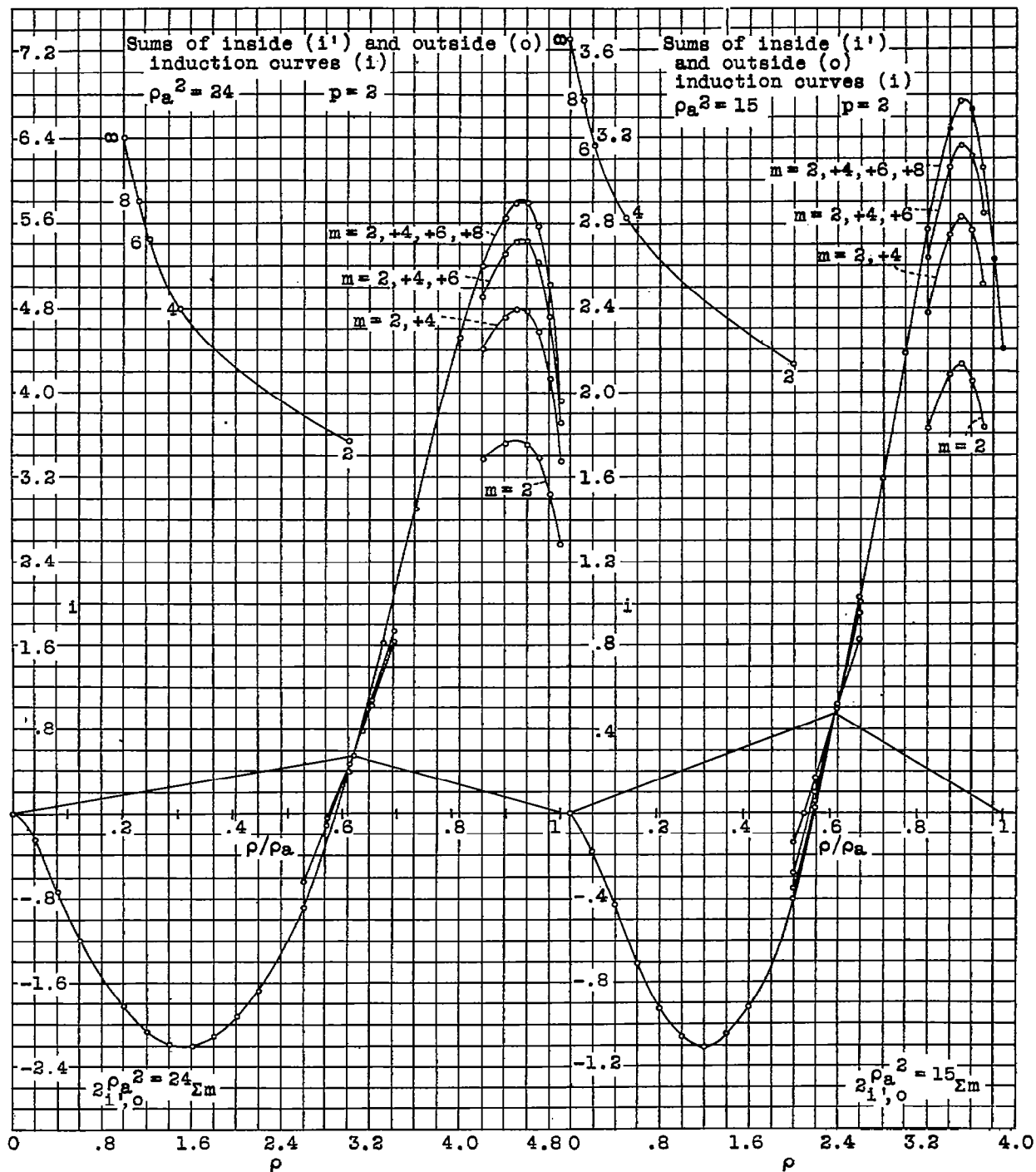


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Figure 24.

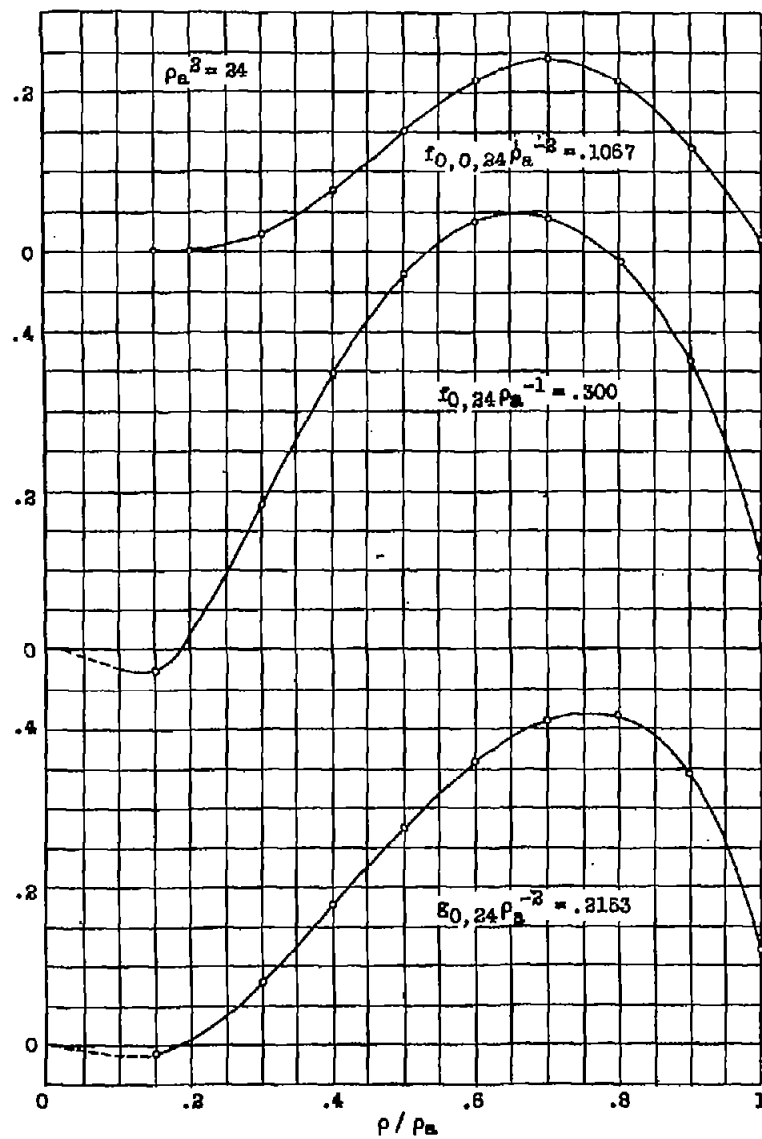


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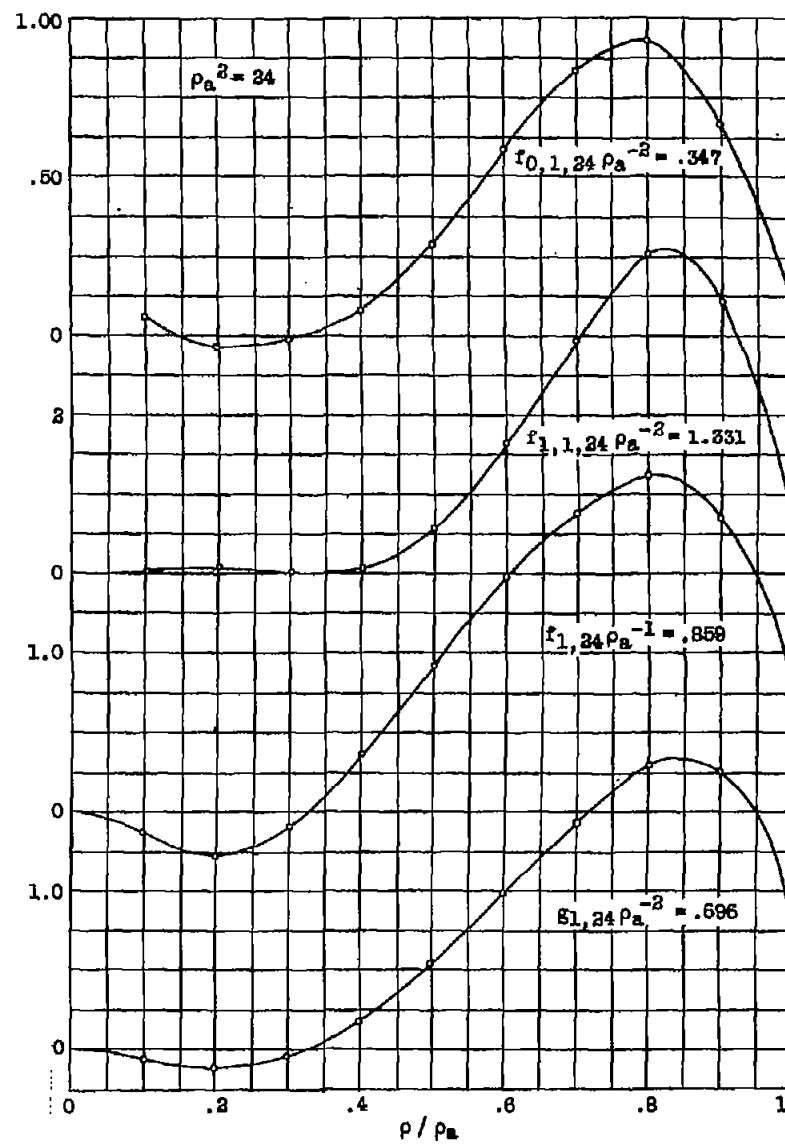


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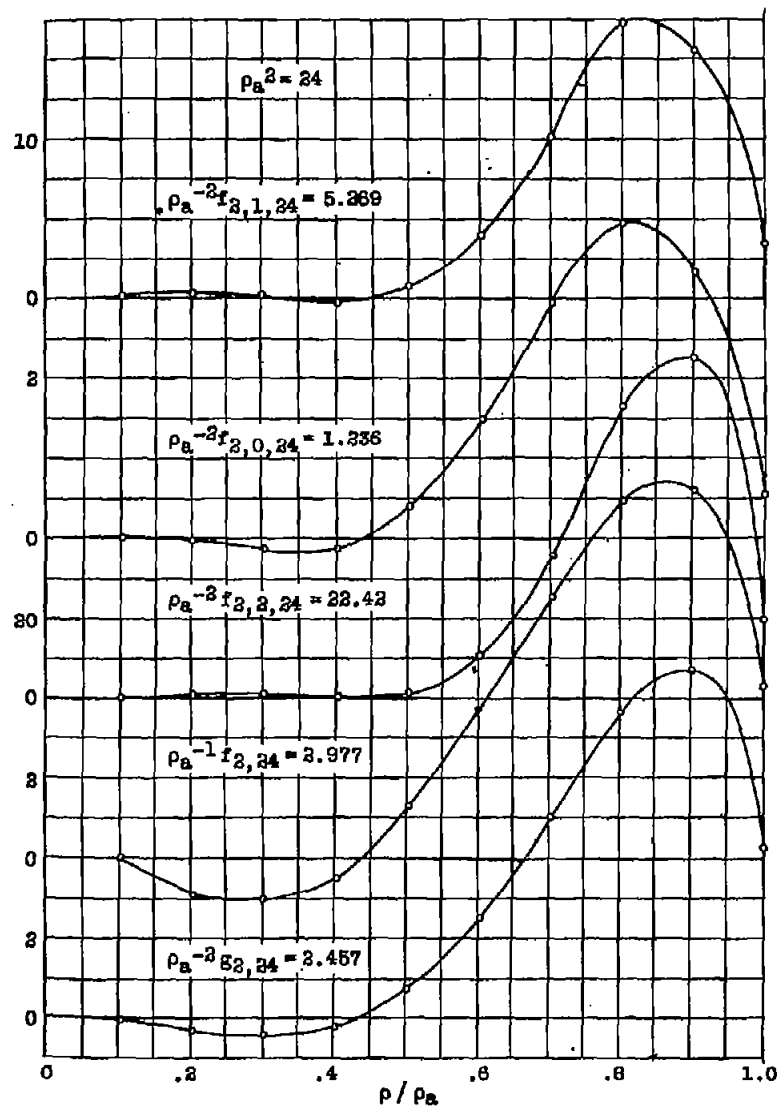


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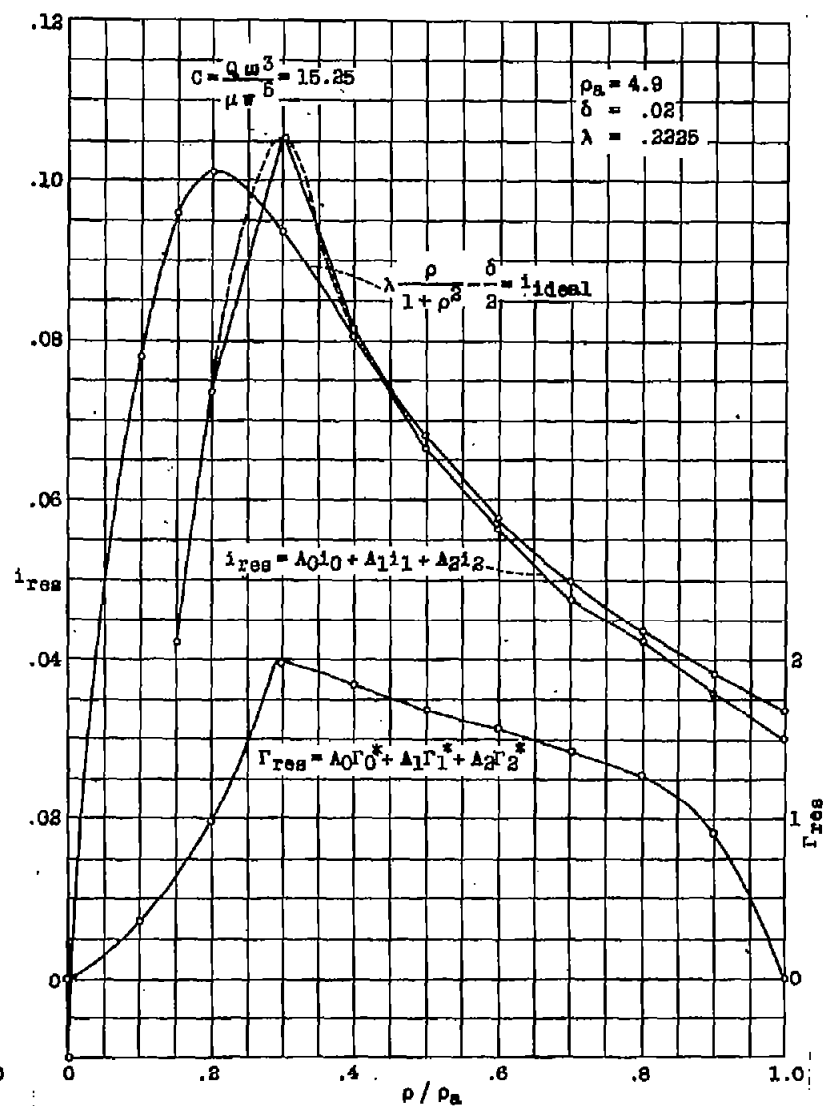


Figure 28.